Chapter 2

Political Compromise and Savings

How does a government controlled by different parties allocate spending intra period and across time? The main characteristic of such a political game is the lack of commitment, the inability to tie the hands of future governments to follow today's incumbent wishes. However, cooperation is efficient. Parties are not continuously in power and would like to smooth their share of government spending.

In this chapter, I present a political economy model of government savings. Two political parties alternate in power every period. The party in power at any period controls the government and decides how to allocate spending this period and how much to save for the future. No party has the ability to commit and at any point in time a party can spend all the income of the government in her own consumption and save nothing for the future. If both parties behave as described, then these strategies are a subgame perfect equilibria. However, parties are long run players in this political game, and they might be expected to coordinate and play more efficient equilibria. I study, then, the set of efficient subgame perfect equilibria.

Two main results are obtained. First, as expected, cooperation is sustainable for low discount rates and low elasticities of intertemporal substitution. They both reduce the incentive to cheat relative to to cooperate. However, differently from previous results, there is no cooperation in cases where the marginal utilities of consumption of the parties in the future are equal to infinity. I show that the set of efficient equilibria can be very sensitive to small parameter changes. Small decreases in the discount rate can discontinuously make cooperation disappear.

In the technical side, the paper shows how to bypass the non-convexity that appears when incentive constraints are dependent on previous play. In the case of power utility and no endowment I propose a technique to compute the equilibrium set numerically and characterize the intra period spending and asset holdings.

These results also highlight the role that illiquid assets might play in generating more efficient allocations. Illiquidity is a useful characteristic because it reduces the temptation of current governments from overconsuming, and allows political parties to smooth spending accross time more easily. So, as in the case of hyperbolic consumers (see Laibson (1994)) there is a demand for illiquidity. This calls into question the efficiency of the "privatization" programs that have been applied in Latin America in the last decade. A government might hold illiquid assets even when their rate of return is lower than alternatives, because these might be the only type of assets that they are not tempted to consume once in power. If liquidating such assets becomes suddenly cheaper, previous allocations that supported cooperation across parties might become unsustainable.

Several papers have analyzed the impact of lack of commitment in political economy models (see for example, Acemoglu and Robinson (2001), Alesina and Tabellini (1990), etc.). This paper extends this literature by showing how the asset holdings of a government are influenced by the political constraints when parties cannot commit to future plays. The set up of the model is close to Dixit, Grossman and Gul (2000). These authors characterize the set of efficient equilibria payoffs in a model where parties alternate in government probabilistically every period. Every period the government receives an endowment which cannot be pledge nor saved, but can be consumed by the parties. In this paper however, the focus is in the role that assets play in a political equilibrium, and the parties are allowed to save or disave. However the savings technology is common to all parties, and the party in power at any period controls all the income received from previous government savings.

The next section explicitly sets up the model.

2.1 Political Compromise and Savings

There are two parties, A and B. They generate utility from the consumption of two public goods (g_t^A and g_t^B) provided every period by the government. Given a sequence of government spending $g = \left\{ \left(g_0^A, g_0^B\right), \left(g_1^A, g_1^B\right), ...\right\}$, the utilities of both parties are given by

$$v_0^A(g) = \sum_{t=0}^{\infty} \beta^t u\left(g_t^A\right)$$
$$v_0^B(g) = \sum_{t=0}^{\infty} \beta^t u\left(g_t^B\right)$$

where $u(x) = x^\theta$ with[1] $\theta \in (0,1)$.

The government can save in a one period bond with a rate of return R. Every period, before deciding on spending and savings, the government receives an endowment y plus the return of the previous period bond holdings. Let a_t be the savings done by the government in period t.

The government at period t faces the following budget constraint

$$y + Ra_{t-1} = a_t + g_t^A + g_t^B$$

I assume that the government cannot borrow, so $a_t \geq 0$ $\forall t$. It is also assume that

Assumption 2.1 *Let* $\beta R = 1$.

With this assumption, in a first best allocation it is always optimal to maintain the asset level constant.

The political structure is described in the next assumption

Assumption 2.2 *In odd periods, party A controls the government. In even periods, party B controls the government. The party that controls the government at some time t chooses the spending allocation at time t and the savings done by the government for next period.*

The game is then as follows. At any time, the party that controls the government has discretion about choosing the savings and spending done by the government in that period.

This game might obviously have multiple equilibria. As we will see, there exist equilibria where parties cooperate and share from government spending every period. There

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[1] θ is restricted to be bigger than zero, because otherwise the punishment of no cooperation is too strong, and first best allocations are always incentive compatible.

are also equilibria where parties spend only on their own goods while in power which is clearly inefficient. We would like to characterize the set of efficient equilibria of this dynamic game. Following Abreu, Pearce and Stachetti (1990), all efficient equilibria can be sustained by the threat of moving to the worst possible subgame perfect equilibrium. We need first to characterize this worst possible equilibrium, and this is what follows.

2.2 Worst Equilibrium

The worst possible equilibrium is supported by the following strategies. The party in power consumes all of the government income and saves nothing. The other party behaves just like that. Given that both parties follow the previous strategy, it is easy to see that it constitutes a subgame perfect equilibrium.

What are the payoffs to the parties in power? Suppose that the party in power receives an asset income of Ra. The party in power spends on her good all of the current government income, that is $y + Ra$, and thereafter consumes 0 every time she is not in power and y every time she is. The utility for the party is then

$$\bar{v}\left(Ra\right) = u\left(y + Ra\right) + \frac{\beta}{1-\beta^2}\left(u\left(0\right) + \beta u\left(y\right)\right) = u\left(y + Ra\right) + \frac{\beta^2}{1-\beta^2}u\left(y\right)$$

Note that this is the minimum payoff that any party with assets holdings of Ra can get in any equilibrium.

What are the dynamics of savings in the worst equilibrium? Parties every period save nothing for the next, $a_t = 0$ for all $t > 0$. The political economy problem does not allow a party today to appropriate the return on the savings she has made. The party in power in the future will use the savings in the consumption of her own goods, from which the party today derives no utility. This inability to tie the hands of the party in power tomorrow, makes the party in power today unwilling to save. This is a result well known and studied in the political economy literature (Alesina and Tabellini (1991), Persson and Svensson (1991)), politicians are impatient and tend to consume too much out of their assets once in power. The objective of this paper is to expand these results for the case when parties are able to sustain reputation, and analyze the role that assets play in a reputational equilibrium. Next section begins to characterize the set of efficient subgame perfect equilibria, equilibria that are sustained by the threat of reverting to the worst equilibrium described above.

2.3 Efficient Subgame Perfect Equilibria

In this section we proceed to characterize the set of efficient equilibria. A couple of definitions follow:

Definition 2.1 *We say that* g *is feasible if* $g_t^A \geq 0$ *and* $g_t^B \geq 0 \ \ \forall t$; *and*

$$\sum_{i=1}^{t} \left(\frac{1}{R}\right)^{i-1} \left(y - \left(g_i^A + g_i^B\right)\right) + Ra_0 \geq 0; \ \ \forall t$$

Definition 2.2 *For any* g*, we say* a_t *to be generated by* g *if*

$$a_t = y + Ra_{t-1} - \left(g_t^A + g_t^B\right)$$

Note that feasibility is equivalent to $a_t \geq 0$.

Definition 2.3 *We say that a feasible* g *is incentive compatible if*

$$\begin{aligned} v_t^A(g) &\geq u^A(y + Ra_{t-1}) + \frac{\beta^2}{1-\beta^2} u(y) \ \ \textit{for all odd } t \\ v_t^B(g) &\geq u^B(y + Ra_{t-1}) + \frac{\beta^2}{1-\beta^2} u(y) \ \ \textit{for all even } t \end{aligned}$$

with a_t *being generated by* g.

Call $F(a_0)$ the set of all g that are feasible with initial assets a_0.

Call $I(a_0)$ the set of all g that are feasible and incentive compatible with initial assets a_0. Clearly $I(a_0) \subset F(a_0)$.

The set of efficient SPE is characterized by the solutions to the following program

$$V(a_0, \psi) : \max_{g \in I(a_0)} v_0^B(g) \qquad \text{subject to } v_0^A(g) \geq \psi \tag{2.1}$$

Where $V(a_0, \psi)$ represents the maximum payoff possible for party B subject to providing a minimum payoff of ψ to party A. Note that in the first period party A is in power.

Before analyzing the program in (2.1), I proceed to characterize the first best allocations of the game (the allocations under committment). This is done in the next section.

2.4 First Best Allocations

A first best allocation solves the following relaxed program

$$V^{FB}\left(a_0, \psi\right) : \max_{g \in F(a_0)} v_0^B\left(g\right) \qquad \text{subject to } v_0^A\left(g\right) \geq \psi$$

The difference is that the program is now restricted only to allocations that are feasible, and incentive compatibility is not taken into account.

Under Assumption 2.1, in a first best, allocation total spending by the government is constant, and both parties receive a constant share of it every period.

Result 2.1 *A first best allocation is characterized by a constant savings decision* $a_t = a_0$*, and* $g_t^A + g_t^B = (R-1)a_0 + y = g^T$ *and* $g_t^A = \alpha g^T$ *; for some* $\alpha \in [0,1]$*.*

It is possible then to index the set of first best allocations by the fraction α of total spending allocated to party A's consumption. The respective payoffs under a first best allocation for the parties are

$$v^A = \frac{u(\alpha g^T)}{1-\beta} \quad \text{and } v^B = \frac{u((1-\alpha)\, g^T)}{1-\beta}$$

When is a first best allocation incentive compatible? It is now known that all first best allocations are index by α, and each party receive a constant payoff throughout. To check incentive compatibility, the payoffs under the first best allocation should be compared to the payoffs (which is also constant) under the worst equilibrium.

2.4.1 Incentive Compatibility of First Best Allocations

Note that in a first best allocation, the payoff to the parties are stationary. For a first best α to be incentive compatible the following inequalities have to hold

$$\begin{aligned} \frac{u(\alpha g^T)}{1-\beta} &\geq u(y + Ra_0) + \frac{\beta^2}{1-\beta^2} u(y) \\ \frac{u\left((1-\alpha)\, g^T\right)}{1-\beta} &\geq u(y + Ra_0) + \frac{\beta^2}{1-\beta^2} u(y) \end{aligned}$$

At any time a party comes to power, she has to receive under the first best allocation more than what she can get by deviating towards the worst equilibrium. Given the symmetry of the model, a first best allocation will be incentive compatible if the first best allocation with $\alpha = \frac{1}{2}$ is.

Lemma 2.1 *If for some asset level a_0, the first best allocation indexed by $\alpha = 1/2$ is not incentive compatible, then no first best allocation for the same asset level will be subgame perfect .*

For the first best allocation with $\alpha = 1/2$ to be a subgame perfect, the following condition has to be satisfied,

$$\frac{u\left(\frac{y+(R-1)a_0}{2}\right)}{1-\beta} \geq u(y+Ra_0) + \frac{\beta^2}{1-\beta^2}u(y)$$

Or equivalentely,

$$H\left(a_0\right) = u\left(\frac{y+(R-1)a_0}{2}\right) - (1-\beta)\,u(y+Ra_0) - \frac{\beta^2}{1+\beta}u(y) \geq 0 \qquad (2.2)$$

The function $H\left(a\right)$ is positive whenever there is enough surplus in a first best allocation

to satisfy the incentive constraints of both parties whenever they come to power.

As $\beta \to 0$, $H\left(a_0\right)$ converges to $u\left(\frac{y+(R-1)a_0}{2}\right) - u(y+Ra_0) < 0$. As $\beta \to 1$, $H\left(a_0\right)$ converges to $u\left(y/2\right) - \frac{u(y)}{2} > 0$ (where the inequality follows by concavity of u). The next result then follows.

Result 2.2 *For any initial asset level a_0 there exists $\underline{\beta}\left(a_0\right)$ with $0 < \underline{\beta}\left(a_0\right) < 1$, such that for any $\beta \geq \underline{\beta}\left(a_0\right)$, there exists a first best allocation that is subgame perfect; and for any $\beta < \underline{\beta}\left(a_0\right)$ no first best allocation is subgame perfect.*

How does $H\left(a_0\right)$ change with a_0? When $a_0 = 0$, $H\left(a_0 = 0\right) = 2^{-\theta} - \frac{1}{1+\beta}$. The function $H\left(a_0\right)$ starts positive then if $2^{\theta}\left(1+\beta\right)^{-1} < 1$. What happens when a_0 go to infinity? We can write

$$H\left(a_0\right) = \left(a_0\right)^{\theta}\left[\left(\frac{y}{2a_0} + \frac{(R-1)}{2}\right)^{\theta} - (1-\beta)\left(\frac{y}{a_0} + R\right)^{\theta} - \frac{\beta^2}{1+\beta}u(\frac{y}{a_0})\right]$$

As a_0 goes to infinity, the cooperation value is growing at a rate of $\left(\frac{(R-1)}{2}\right)^{\theta}$ and the temptation at a rate $(1-\beta)\,R^{\theta}$. So if $\left(\frac{R-1}{2}\right)^{\theta} > (1-\beta)\,R^{\theta} \Rightarrow$

$$2^{\theta}\,(1-\beta)^{1-\theta} < 1\ , \tag{2.3}$$

eventually the cooperation gains from having more assets will overcome the increase in the value of the worst equilibrium, and first allocations will be incentive compatible for high enough level of assets. If the condition (2.3) holds with opposite sign, then as a_0 increases, the value from non-cooperation will overtake the benefits, and no first best allocation will be incentive compatible.

Figure 1 plots $H\,(a_0)$ for different parameter values.

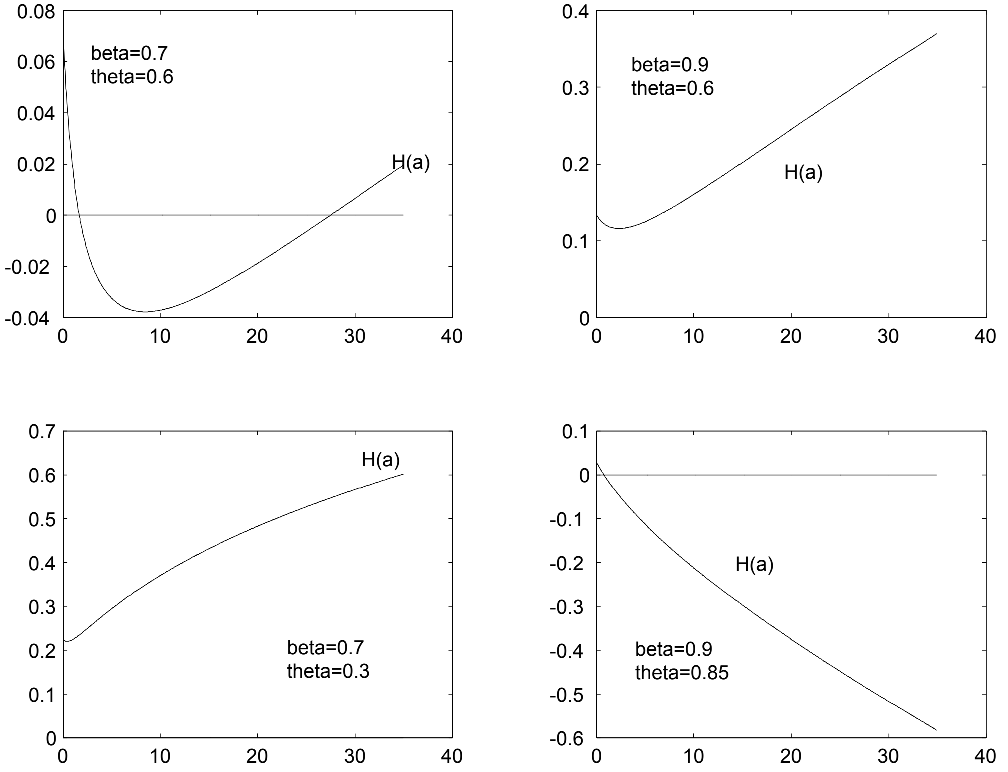

Figure 2-1: Is First Best Incentive Compatible?

As it is possible to see from the graph, the relation is not monotonic. In the top left panel, for example, there is an intermediate level of assets for which no first best allocation is incentive compatible, even when there are incentive compatible first best allocations for low and high asset levels.

2.5 Efficient Subgame Perfect Equilibria

In this section, I study the set of incentive compatible and feasible allocations that are efficient.

Let a_0 be the initial amount of asset holdings. Let ψ be the utility level promised to party A(in power in period 1). Then, the set of efficient SPE is parameterized by a_0 and ψ and is the set of solutions to the following program :

$P(a_0, \psi)$:

$\max_{\{g^A, g^B, a\}} E_0 \sum_{i=0}^{\infty} \beta^i u^B(g^B_{i+1})$ subject to

$$\sum_{i=0}^{\infty} \beta^i u^A(g^A_{i+1}) \ \geq \psi$$

$$a_t \geq 0; \ g^A_t \geq 0; \ g^B_t \geq 0; \ \forall t$$

$$y + Ra_{t-1} = g^A_t + g^B_t + a_t; \ \forall t$$

and subject to the incentive constraints,

$$\sum_{i=0}^{\infty} \beta^i u\left(g^A_{t+i}\right) \geq u\left(y + Ra_{t-1}\right) + \frac{\beta^2}{1-\beta^2} u\left(y\right) \text{ ; for all even } t \geq 2 \tag{2.4}$$

$$\sum_{i=0}^{\infty} \beta^i u\left(g^B_{t+1}\right) \geq u\left(y + Ra_{t-1}\right) + \frac{\beta^2}{1-\beta^2} u\left(y\right) \text{ ; for all odd } t \geq 2 \tag{2.5}$$

The first problem is that the constraint set is not convex. Because previous choices of asset holdings affect the incentive constraint in the next period, choice variables of the past will appear as a convex function on the left hand side of the incentive constraint. The constraint set is, then, not convex. Hence, even if the value function were differentiable first order conditions won't be sufficient for optimality.

Let $V(a, \psi)$ be the maximum value attainable to the party **not** in power today when the party in power has been promised a utility level of ψ and the savings done yesterday were equal to a. Define the domain constraint of $V(a, \psi)$ as a correspondence $D(a)$ such that $D(a)$ is an interval of the real line :

$$D(a) = \left[\underline{\psi}(a), \bar{\psi}(a)\right]$$

where $\underline{\psi}(a) = u(y + Ra) + \frac{\beta^2}{1-\beta^2}u(y)$. Let the operator T^V be defined in the following way:

$$T^V \{V(a, \psi), D(a)\} = \max_{g_1, g_2, a', \psi'} \left\{u\left(g^1\right) + \beta\psi'\right\} \tag{2.6}$$

subject to:

$$u(g_2) + \beta V(a', \psi') \geq \psi \tag{2.7}$$

$$y + Ra - g_1 - g_2 - a' = 0 \tag{2.8}$$

$$a' \geq 0 \tag{2.9}$$

$$\psi' \in D(a') \tag{2.10}$$

where condition (2.7) is the promise keeping constraint: the utility delivered to the party in power today cannot be smaller than what was promised to her yesterday. Condition (2.8) is the budget constraint of the government today. Condition (2.9) is the non-borrowing constraint of the government. And condition (2.10) is the incentive constraint for the party in power tomorrow and a feasibility constraint on the promise utility (whatever is promised for tomorrow cannot be too high, otherwise it won't be feasible). A prime on top denotes next period choices.

And let the operator T^D be

$$T^D [V(a, \psi), D(a)] = [\underline{\psi}(a), T\bar{\psi}(a)]$$

where

$$T\bar{\psi}(a) = \max_{\substack{0 \leq g_2 \leq Ra+y \\ \tilde{\psi}' \in D(y+Ra-g_2)}} \left\{u(g_2) + \beta V\left(y + Ra - g_2, \tilde{\psi}'\right)\right\} \tag{2.11}$$

Remark: *The operator* T^V *maps a function* V *(with a domain* D*) into a new function* $T^V V$ *that is the maximum value attainable to the party not in power subject to all the relevant constraints. The operator* T^D *maps a domain of* V *into a new domain* $T^D D$ *which basically reads that whatever is promised tomorrow to the party in power has to be higher than her autarky value tomorrow and smaller than the maximum promise that* *can be made,* (2.11)*. The solution to our original program is the biggest fixed point of* $\left(T^V, T^D\right)$*.*

The recursive formulation can be interpreted as follows. The value offered to the party not in power today -given a promised utility value to the party in power today and an asset level- is obtained by maximizing over a new non-negative asset level, consumption levels and a new promised utility such that the party in power today obtains at least a utility level equal to her promised utility (2.7), the budget constraint is satisfied (2.8), and the party in power tomorrow cannot be promised a utility level below the worst equilibrium payoff (2.10), given the asset level selected for tomorrow.

It's not known whether the value function is concave or differentiable, but it is still possible to derive a few interesting results.

Proposition 2.1 *If the incentive constraint at time* $t+1$ *is not binding and* $a_t > 0$*, then* $g^A_{t+1} = g^A_t$ *and* $g^B_{t+1} = g^B_t$

Proof. Suppose w.lo.g. that party A was in power in period $t+1$ and that $g^A_{t+1} \neq g^A_t$. Then you can bring g^A_t closer to g^A_{t+1} without affecting a_{t+1}, g^B_t and g^B_{t+1} by changing the amount saved at time t (a_t). This change has no effect in the utility of party B at time t and $t+1$, and increases the utility of party A by $|u'\left(g^A_{t+1}\right) - u'\left(g^A_t\right)|$, without affecting her incentive constraint tomorrow (it was slack). The incentive constraint of party B was not affected by this change (her utility remained the same, and the assets held at time t (a_{t-1}) did not change). If the incentive constraint at time $t+1$ is not binding, it has to be the case then that in an efficient allocation $g^A_t = g^A_{t+1}$. If $g^B_t \neq g^B_{t+1}$ a similar argument applies, a marginal movement of the assets at time t will increase the utility of party B at time t (making her incentive constraint slack) and will not tight the incentive constraint of party A at time $t+1$ (which was slack). ■

This characterization says that if the incentive constraint is not binding at some period, then it is efficient to maintain the spending allocation between this period and the previous one. If incentive constraints never bind, then the following is clear,

Corollary 2.1 *If in an efficient incentive compatible allocation* g *no incentive constraint ever binds, then* g *is a first best allocation.*

What about if the incentive constraints are binding?

Proposition 2.2 *If party i is in power at time $t+1$ and her incentive constraint is binding, then $g_{t+1}^{i} \geq g_{t}^{i}$*

Proof. Let w.l.o.g. $i = A$. The proof proceeds by contradiction. Suppose that $g_{t+1}^{A} < g_{t}^{A}$. Then it is possible to increase the savings done at time t (a_t), by one unit, and assign all the returns (R) to party A's consumption at time $t+1$. This does not affects party B's payoff at time t, nor her incentive constraint at time t (a_{t-1} hasn't change). However, party A's payoff for deviating at time $t+1$ increased by $Ru'(y + Ra_t)$ while her utility from cooperating at time $t+1$ increased by $Ru'\left(g_{t+1}^{A}\right)$. Given that $y + Ra_t > g_{t+1}^{A}$ the benefits from cooperation increased more than the costs at time $t+1$, so the incentive constraint of party A at time $t+1$ is holding. Note also that the utility of party A at time t is now higher by an amount $u'\left(g_{t+1}^{A}\right) - u'\left(g_{t}^{A}\right) > 0$. This is then a more efficient allocation than the initial one, a contradiction. ■

A similar proposition can be proved for the consumption behavior of the party not in power at the time an incentive constraint binds,

Proposition 2.3 *Let $a_t > 0$. If party i is in power at $t+1$ and her incentive constraint is binding, then $g_{t+1}^{-i} \leq g_{t}^{-i}$.*

Proof. Let w.l.o.g. $i = A$. The proof proceeds by contradiction. Suppose now that $g_{t+1}^{B} > g_{t}^{B}$. Then you can reduce the savings done at time t, a_t, without affecting a_{t+1}, g_{t+1}^{A} and g_{t}^{A}, this reduces the assets holdings at time $t+1$, reducing the benefit from deviating of party A while the benefit from cooperation at time $t+1$ has not changed. The incentive constraint for party A at time $t+1$ is now slack. However, the decrease in the savings increases g_{t}^{B} and decreases g_{t+1}^{B}, at a rate R. The utility of party B at time t is then increased. This also makes her incentive constraint slack, the benefits from cooperation increased, while the value of deviating remained the same (a_{t-1} hasn't changed). So, this movement is incentive compatible and has increased the utility of party B while keeping party A's payoff constant. Then, an allocation with $g_{t+1}^{B} > g_{t}^{B}$ cannot be efficient. ■

The previous propositions characterize the behavior of government spending across parties with the changes of political power and whether the incentive constraint are binding or not. Whenever an incentive constrain binds, the consumption allocated to the party in power at that time will increase and the consumption allocated to the party out of power will decrease. The proofs have shown how assets are playing a double role in an efficient allocation. Assets help to smooth consumption across time, but at the same time, they change the incentive constraints of the parties in power by directly affecting the value from moving to the worst equilibrium at any time. It has proven difficult to analyze the behavior of the asset holdings for the general case. Next section characterizes the efficient set when there is no endowment ($y = 0$). The power utility structure is exploited very heavily throughout that section. As will be clear, with no endowment and power utility, the program becomes homogenous, which allow me to bypass the non-convexity of the constraint set and characterize the efficient equilibria.

2.6 The Savings Game with No Endowment

In this section the case with no endowment is analyzed.

2.6.1 First Best Allocations and Incentive Compatibility

Result 2.3 *If there is no endowment ($y = 0$), there exists a first best allocation that is subgame perfect if and only if*

$$1 \geq (1-\beta)^{1-\theta} 2^{\theta} \tag{2.12}$$

Remark: *Note that this condition is equivalent to condition* (2.3)*, that apply in the limit as a_0 approaches infinity. In this case the endowment has no relevance, and it is not surprising that the condition for sustainability of the first best is the same as in the no-endowment case.*

Note also that the condition for the existence of an incentive compatible first best allocation with no endowment is independent of the level of initial assets a_0. The reason is that given power utility, the utility under the worst equilibrium and the utility under a first best allocation is proportional to $(a_0)^{\theta}$. So, if for some asset level there exits an incentive compatible first best allocation, then it exists for any asset level. Note also that as θ goes to zero, the inequality is more likely to hold; and the opposite occurs as θ goes to 1. The intuition for that result is that as $\theta \to 1$, the elasticity of intertemporal substitution of the parties increases and the desire to smooth spending across time is reduced. This makes cooperation harder to sustain. In the limit when $\theta = 1$, the parties are linear, and there is no gain from intertemporal smoothing.

2.6.2 The Efficient Subgame Perfect Frontier

The first thing to notice is the following.

Proposition 2.4 *In the game with no endowment, if in an incentive compatible allocation for some t, $a_t = 0$, then this allocation is the worst equilibrium.*

If at some point in time, savings are zero, then the party at that time in power at time will spend all the assets on her own good (by incentive compatibility). This implies then that the party in power a period before will not save either and will consume all the assets on her own good, and so on. The allocation is then the autarky equilibrium.

Let us redefine the value function as $V(a,\psi) = \hat{V}(a,\tilde{a})$, where $u(R\tilde{a}) = \psi$. Using the fact that there is no endowment ($y = 0$), the operator T (2.6) is an operator in the value function, T^V

$$T^V\left[\hat{V}(a,\tilde{a}), \hat{D}(a)\right] = \max_{\{a',\tilde{a}',g_1,g_2\}} \{u(g_1) + \beta u(R\tilde{a}')\}$$

subject to:

$$u(g_2) + \beta\hat{V}(a',\tilde{a}') \geq u(R\tilde{a}) \tag{2.13}$$

$$Ra - g_1 - g_2 = a' \tag{2.14}$$

$$a' \geq 0 \tag{2.15}$$

$$\tilde{a}' \in \hat{D}(a') \tag{2.16}$$

and an operator T^D in the domain constraint,

$$T^D\left[\hat{V}(a,\tilde{a}), \hat{D}(a)\right] = [a, \bar{\chi}(a)]$$

where

$$\bar{\chi}(a) = \frac{1}{R}u^{-1}\left(\max_{\substack{0\leq g_2\leq Ra \\ \tilde{a}'\in\hat{D}(Ra-g_2)}} \left\{u(g_2) + \beta\hat{V}(Ra-g_2,\tilde{a}')\right\}\right) \geq a$$

We are interested in the biggest fixed point of T, $V^*(a,\tilde{a}), D^*(a)$ such that $V^* = T^V V^*$ and $D^* = T^D D^*$.

We say that a correspondence $D(a)$ is homogenous if $\lambda\hat{D}(a) = \hat{D}(\lambda a)$ for λ positive.

Lemma 2.2 *The operator T is such that for all $\hat{V}(a, \tilde{a})$ continuous and homogeneous of degree θ in the associated domain $\hat{D}$ homogenous and compact valued we have that*

- *$T^V \hat{V}$ is homogenous of degree θ*
- *$T^D \hat{D}$ is homogenous.*

Proof. The domain operator is homogenous clearly if $\hat{V}$ is homogenous of degree θ. Let $g_1(a, \tilde{a})$,$g_2(a, \tilde{a})$, $a'(a, \tilde{a})$ and $\tilde{a}'(a, \tilde{a})$ be the policy functions that solve $T^V V(a, \tilde{a})$, then it is easy to see that $\lambda g_1(a, \tilde{a}), \lambda g_2(a, \tilde{a})$,$\lambda a'(a, \tilde{a})$ and $\lambda \tilde{a}'(a, \tilde{a})$ satisfy the constraint set for $T^V V(\lambda a, \lambda \tilde{a})$ with λ positive. This implies then that $\lambda^\theta T^V V(a, \tilde{a}) \leq T^V V(\lambda a, \lambda \tilde{a})$. Given that the choice of λ is arbitrary (we could have chosen $1/\lambda$) then $\lambda^\theta T^V V(a, \tilde{a}) = T^V V(\lambda a, \lambda \tilde{a})$. ■

Lemma 2.3 *The operator T is monotone. For any two value functions V_1 and V_2 with respective domains D_1 and D_2 then if*

$$V_1 \leq V_2 \text{ and } D_1 \subset D_2 \Rightarrow T^V V_1 \leq T^V V_2 \text{ and } T^D D_1 \subset T^D D_2$$

Theorem 2.1 *The value function $V^*(a, \tilde{a})$ is homogenous of degree θ.*

Proof. Start with $V_0(a, \tilde{a})$ of the first best allocations with $D_0(a) = \left[a, a\left(\frac{R}{R-1}\right)^{\frac{1-\theta}{\theta}}\right]$. In the first iteration, clearly we have that $T^V V_0 \leq V_0$ and $D_0 \subseteq T^D D_0$. Now, because $V_0(a, \tilde{a})$ is homogenous of degree θ, then $T^V V_0$ is homogenous as well (the same applies for the domain operator). The value of V^* is such that $V^* \leq V_0$, so monotonicity implies that $\left(T^V\right)^n V^* = V^* \leq \left(T^V\right)^n V_0$, (the same for D_0). We have sequences of $\left(T^V\right)^n V_0$ and $\left(T^D\right)^n D_0$ that are monotonically decreasing, and are always bigger than V^* and D^* so, the sequences have to converge. Given that they converge to a fixed point, it has to be the case that $\lim_{n\to\infty} \left(T^V\right)^n V_0 = V^*$ ($[V^*, D^*]$ is the biggest fixed point). Because $\left(T^V\right)^n V_0$ are homogenous for all n, we have that V^* is homogenous. ■

The interesting cases to study are when the incentive constraints are binding. From now on, the following assumption is made

Assumption 2.3 *There is no incentive compatible first best allocation:*

$$1 < (1-\beta)^{1-\theta}\, 2^{\theta}$$

The incentive constraints will be binding.

Theorem 2.2 *If no first best allocation is incentive compatible, then the incentive constraints are always binding.*

To proof Theorem 2.2, the following lemma is used.

Lemma 2.4 *If no first best allocation is incentive compatible, then an incentive compatible allocation cannot provide both parties at the same time with a utility level higher than their autarky value.*

Proof. By Assumption (2.3), there is no first best allocation that can give each party at the same time a utility level higher than their autarky value. Now, first best allocations give the maximum amount of utility to one party constrained to providing certain value to the other. Given a utility value for one party, any other allocation will provide a utility level to the other party smaller than the corresponding first best allocation. Now, given that no first best allocation was incentive compatible, that implied that at any time it was not possible in a first best allocation to provide both parties with utility levels higher than their autarky values, then no other allocation will do it. ■

This lemma says that if no first best allocation is incentive compatible, then at any time t if one party receives a utility level higher or equal to $u\left(Ra_t\right)$, then other party has to receive a value strictly smaller than $u\left(Ra_t\right)$ (Notice that the party who receives less than $u\left(Ra_t\right)$ is the party out of power at time t, otherwise her incentive constraint won't be holding).

Now, it is possible to proof Theorem 2.2.

Proof of Theorem 2.2. First, it is shown that if the incentive constraint is not binding at time $t+1$, it cannot be binding at time $t+2$. Suppose not, that the incentive constraint is binding at time $t+2$ and it was not binding at $t+1$. Let w.l.o.g. party A be in power at time t. Then by Proposition 2.1, $g_{t+1}^{A} = g_t^{A} = g_A$ and $g_{t+1}^{B} = g_t^{B} = g_B$. At time $t+2$, party A is receiving a utility level equal to her autarky value (the incentive constraint is binding), so $V^A_{t+2} = u\left(Ra_{t+2}\right)$. At time $t+1$, by Lemma 2.4 the utility to party A $\left(u\left(g_A\right)+\beta V^A_{t+2}\right)$ is smaller than her autarky value and the utility to party B is higher than her autarky value (her incentive constraint is not binding at $t+1$), so

$$u\left(g_A\right)+\beta u\left(Ra_{t+2}\right) < u\left(Ra_{t+1}\right) < u\left(g_B\right)+\beta V^B_{t+2} \tag{2.17}$$

Where V^B_{t+2} is the value party B receives in period $t+2$. In period $t+2$, party B is out of power, and receives by Lemma 2.4 a value smaller than her autarky value, $V^B_{t+2} < u\left(Ra_{t+2}\right)$. Plugging into (2.17),

$$\begin{aligned} u\left(g_A\right)+\beta u\left(Ra_{t+2}\right) &< u\left(g_B\right)+\beta V^B_{t+2} < u\left(g_B\right)+\beta u\left(Ra_{t+2}\right) \\ u\left(g_A\right)-u\left(g_B\right) &< 0 \Rightarrow \end{aligned}$$

$$g_A < g_B \tag{2.18}$$

The incentive constraint of party A is binding at time $t+2$, so the incentive constraint at time t for party A implies that

$$V^A_t = u\left(g_A\right)+\beta u\left(g_A\right)+\beta^2 u\left(Ra_{t+2}\right) \geq u\left(Ra_t\right) \tag{2.19}$$

By Lemma 2.4, party B in period t, receives a utility value that has to be smaller than her autarky value. So $u\left(g_B\right)+\beta V^B_{t+1} \leq u\left(Ra_t\right)$. But, $V^B_{t+1} > u\left(Ra_{t+1}\right)$ (her incentive constraint is not binding at time $t+1$), so

$$u\left(g_B\right)+\beta u\left(Ra_{t+1}\right) < u\left(Ra_t\right) \tag{2.20}$$

Then by (2.18), (2.19) and (2.20) it is obtained

$$\frac{u\left(Ra_t\right)-\beta^2 u\left(Ra_{t+2}\right)}{1+\beta} \leq u\left(g_A\right) < u\left(g_B\right) < u\left(Ra_t\right)-\beta u\left(Ra_{t+1}\right)$$

or equivalently,

$$\beta\left(1+\beta\right)u\left(Ra_{t+1}\right)-\beta u\left(Ra_t\right)-\beta^2 u\left(Ra_{t+2}\right) < 0 \tag{2.21}$$

Let $g^T = g_A + g_B$. Then, the following relation holds

$$\begin{aligned} R^2 a_t - R g^T &= R a_{t+1} \\ R^3 a_t - R^2 g^T - R g^T &= R a_{t+2} \end{aligned}$$

Let $J\left(g^T\right) = (1+\beta)\,\beta u\left(R^2 a_t - R g^T\right) - \beta u\left(R a_t\right) - \beta^2 u\left(R^3 a_t - R^2 g^T - R g^T\right)$. Minimizing J

$$g^{T*} = \left\{\arg\min J\left(g^T\right)\right\} = [R-1]\, a_t$$

Plugging back into the value of $J\left(g^T\right)$, $J\left(g^{T*}\right) = 0$. So, $\min_{g^T}\left[J\left(g^T\right)\right] = 0 \Rightarrow J\left(g^T\right) \geq 0$ for all feasible g^T, and (2.21) can not hold. So, if an incentive constraint is not binding at time $t+1$, then is not binding at time $t+2$ and by induction, is never binding for any $\tau > t+1$. The allocation starting from time $t+1$ is then a first best allocation, but this is a contradiction, because no first best allocation is incentive compatible by Assumption (2.3). ■

From Theorem 2.2 it follows then that $u\left(R\tilde{a}'\right) = u\left(Ra'\right) \Leftrightarrow \tilde{a}' = a'$. Using the homogeneity of the value function, the promised utility constraint can be rewritten as $u\left(g_2\right) + \beta\left(a'\right)^{\theta}\hat{V}(1,1) \geq u\left(R\tilde{a}\right)$ and the program under a binding incentive constraint is now

$$\hat{V}\left(a,\tilde{a}\right) = \max \ \left\{u\left(g_1\right) + \beta u\left(Ra'\right)\right\} \tag{2.22}$$

subject to :

$$\begin{aligned} u\left(g_2\right) + \phi\beta\left(a'\right)^{\theta} &\geq u\left(R\tilde{a}\right) \\ Ra - g_1 - g_2 &= a' \\ a' &\geq 0 \end{aligned}$$

where $\phi = \hat{V}(1,1)$.

It is possible to see that in program (2.22) (when incentive constraints are binding) the policy functions are proportional to the initial asset level a. Because from the second period forward, the value function is always evaluated at $V\left(a',a'\right)$, two different programs will have optimal allocations with the same ratios g_1/a and g_2/a from period 2 onwards. Knowing $V(a,a) = \phi a^{\theta}$ will be enough to characterize the Pareto frontier of the subgame perfect equilibria. The problem has been reduced from two dimensions, to just one. It is necessary now to compute ϕ.

What is the value of ϕ? Suppose there is one unit of the asset and party A is in power. Let $g_t^A = (1-\beta)^{\frac{1}{\theta}} R$. This constant path of consumption delivers a utility to party A equal to $u(R)$. From the budget constraint it is possible to compute the associated consumption allocation for party B subject to a constant savings of one unit, $g_t^B = (R-1)\left[1 - \left(\frac{R-1}{R}\right)^{\frac{1-\theta}{\theta}}\right] > 0$. The utility level generated by g_t^B, $\frac{u\left(g_t^B\right)}{1-\beta}$, is the maximum utility that could be delivered to party B constrained to providing a utility level of $u(R)$ to party A. However, this allocation is not incentive compatible (no first best allocation ever is). So the value to party B, $\hat{V}(1,1) \equiv \phi$, cannot be as high as $\frac{u\left(g_t^B\right)}{1-\beta}$. The next lemma follows,

Lemma 2.5 *When no first best allocation is incentive compatible, then* ϕ *is strictly smaller than* $\bar{\phi} \equiv R^\theta \left[\left(\frac{R}{R-1}\right)^{\frac{1-\theta}{\theta}} - 1\right]^\theta < R^\theta$.

The behavior of the asset level can also be characterized.

Notice that as $\theta \to 1$, $\bar{\phi} \to 0$, and hence $\phi \to 0$, $g_2(a) \to Ra$ and $a'/a \to 0$. In the limit as θ converges to one, the unique equilibrium is autarky. This is not surprising, as the elasticity of intertemporal substitution goes to infinity, there are no gains from trade, and cooperation is not incentive compatible.

For the second period forward, $\tilde{a} = a$. The problem is

$$\max_{\substack{Ra = a' + g_1 + g_2 \\ a' \geq 0}} \left\{ u(g_1) + \beta R^\theta u(a') + \kappa \left(u(g_2) + \phi \beta (a')^\theta \right) \right\} \tag{2.23}$$

where κ is the lagrange multiplier of the promise keeping constraint, with $\kappa > 0$[2]. This is a convex program, and first order conditions will be sufficient for optimality. I will first study the problem for any given $\kappa > 0$, and compute equilibrium characteristics that will have to hold for all κ.

Taking the first order conditions (let μ be the lagrange multiplier on the budget constraint) $u'(g_1) = \mu = \kappa u'(g_2) = \left(\beta R^\theta + \kappa \phi \beta\right) u'(a')$. Solving out for the spending

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[2] Under Assumption 3, the promise keeping constraint is binding in any fixed point of program (2.22) for $\tilde{a} = a$. The proof follows by contradiction. Suppose not, then if in a fixed point the promise keeping constraint is not binding, $g_2 = 0$. Solving for $\phi a^\theta = \max_{0 \leq g_1 \leq (Ra - a')} [u(g_1) + \beta u(Ra')]$, it is obtained $\phi = \frac{R^{2\theta}}{(1+R)^{\theta-1}}$. But, given that the promise keeping is not binding, $\phi > R(1+R)^\theta$. Using the computed value of ϕ, it has to be the case that $1 > \left(\frac{1+R}{R}\right)^{(2\theta-1)} R$. But $(\frac{1+R}{R})^{(2\theta-1)} R \geq 2^\theta (\frac{1+R}{R})^{(\theta-1)} R \geq 2^\theta (1-\beta)^{(1-\theta)} > 1$ where the last step follows from assumption 3. This is a contradiction then and the promise keeping constraint is binding in a fixed point for $\tilde{a} = a$.

amounts as a function of a'

$$g_1 = \left[\beta R^{\theta} + \kappa\phi\beta\right]^{\frac{1}{\theta-1}} a', g_2 = \left[\frac{\beta R^{\theta} + \kappa\phi\beta}{\kappa}\right]^{\frac{1}{\theta-1}} a' \tag{2.24}$$

Using the budget to solve for a',

$$Ra = a' + g_1 + g_2 = a'\left(1 + \left[\beta R^{\theta} + \kappa\phi\beta\right]^{\frac{1}{\theta-1}} + \left[\frac{\beta R^{\theta} + \kappa\phi\beta}{\kappa}\right]^{\frac{1}{\theta-1}}\right)$$

Putting $\beta R = 1$, and solving out for the ratio of assets

$$\frac{a'}{a} = \frac{R}{1 + \frac{1+\kappa^{\frac{1}{1-\theta}}}{\left(1+\kappa\phi R^{-\theta}\right)^{\frac{1}{1-\theta}}} R} \tag{2.25}$$

Let $h(\kappa) \equiv \left(\frac{1+\kappa^{\frac{1}{1-\theta}}}{\left(1+\kappa\phi R^{-\theta}\right)^{\frac{1}{1-\theta}}}\right)$. I will compute the minimum value of $h(\kappa)$ as a function of κ. This can then be used as an upper bound for a'/a.

Taking derivatives of $h(\kappa)$ with respect to κ,

$$\frac{\partial h(\kappa)}{\partial\kappa} = \frac{\frac{1}{\theta-1}\kappa^{\frac{\theta}{1-\theta}}\left(1+\kappa\phi R^{-\theta}\right)^{\frac{1}{1-\theta}} - \frac{1}{\theta-1}\left(1+\kappa\phi R^{-\theta}\right)^{\frac{\theta}{1-\theta}}\phi R^{-\theta}\left(1+\kappa^{\frac{1}{\theta-1}}\right)}{\left(\left(1+\kappa\phi R^{-\theta}\right)^{\frac{1}{1-\theta}}\right)^2}$$

The sign of $\frac{\partial h(\kappa)}{\partial\kappa}$ is equal to the sign of $\kappa - \left[\phi R^{-\theta}\right]^{\frac{1-\theta}{\theta}}$. When $\kappa < \left[\phi R^{-\theta}\right]^{\frac{1-\theta}{\theta}}$, $h(\kappa)$ is decreasing, and when $\kappa > \left[\phi R^{-\theta}\right]^{\frac{1-\theta}{\theta}}$, $h(\kappa)$ is increasing. Then, $h(\kappa)$ is minimized at $\left[\phi R^{-\theta}\right]^{\frac{1-\theta}{\theta}}$. An upper bound on $\frac{a'}{a}$ is

$$\frac{a'}{a} < \frac{R}{1 + h\left(\left[\phi R^{-\theta}\right]^{\frac{1-\theta}{\theta}}\right) R} = \frac{R}{1 + \frac{1+\left[\phi R^{-\theta}\right]^{\frac{1}{\theta}}}{\left(1+\left[\phi R^{-\theta}\right]^{\frac{1}{\theta}}\right)^{\frac{1}{1-\theta}}} R} < 1$$

where the last inequality holds because $\phi < \bar{\phi} \equiv R^{\theta}\left[\frac{R^{\frac{1-\theta}{\theta}}}{(R-1)^{\frac{1-\theta}{\theta}}} - 1\right]^{\theta}$. For all κ, we have that $a' < a$. The following theorem is thus proved,

Theorem 2.3 *In the case of no endowment when no first best allocation is incentive* *compatible, if in an efficient allocation* $a_t > 0$ *for some* $t \geq 1$*, then the level of assets holdings decreases at a constant rate from period 2 onwards.*

If there are positive savings done in an efficient allocation, assets are decreasing continuously towards zero. The intuition is the following. If in an incentive compatible allocation, a constant level of assets is maintained, then the ratio of g_1/g_2 is going to be low. An incentive compatible allocation would have to provide too much consumption to the party in power to keep her from consuming all the asset holdings, given that the asset holdings are going to be high in the future, the same applies for the party in power tomorrow. Given that there is party turnover, the consumption allocated to a given party will vary widely, according to whether she is in power or not. Now, by reducing the asset level, total consumption is higher, and is possible to achieve a higher g_1/g_2 ratio. Now, when parties alternate in power, the changes in their consumption paths are less drastic than before, but have a decreasing slope (less is saved for the future). This trade-off between asset efficiency and the sharing of consumption in the power utility case with no endowment is stronger in the need for consumption sharing, and asset holdings are reduced in an efficient allocation.

I can also characterize the rest of the efficient frontier. Let $a'(a,\tilde{a})$, $g_1(a,\tilde{a})$ and $g_2(a,\tilde{a})$ be the optimal policies with initial asset level a and promised utility $u(R\tilde{a})$. Then, the following holds,

Proposition 2.5 *For any* a*, as the utility level promised to the party in power* $(\tilde{a})$ *increases (in the domain), then*

$$\begin{array}{l} a'(a,\tilde{a}) \ \textit{decreases} \\ \dfrac{g_1(a,\tilde{a})}{g_2(a,\tilde{a})} \ \textit{decreases} \end{array}$$

Proof. To proof this proposition, it is first shown that $\kappa > 1$. To see this, note that Proposition 2.2 implies that $g_1(a,a) \leq g_2(a',a') = \frac{a'(a,a)}{a} g_2(a,a)$. Given that $a'/a < 1$, then $g_2 > g_1$. Using (2.24), it is obtained that $\kappa^{\frac{1}{1-\theta}} > 1$, which implies that $\kappa > 1$. Increasing $\tilde{a}$ is equivalent to increase κ in problem (2.23) -a tightening of the promised keeping constraint- and given that $\kappa > 1 > \left(\phi R^{-\theta}\right)^{\frac{1-\theta}{\theta}}$, it implies by (2.25) that a' is decreasing in $\tilde{a}$ for a given a. The fact that g_1/g_2 decreases is clear by (2.24). ■

It is possible to characterize the rest of the model by solving for ϕ and κ. From the promise keeping constraint,

$$(g_2)^{\theta} + \phi\beta\left(a'/a\right)^{\theta} = R^{\theta}$$

And from the value function definition

$$\hat{V}\left(1,1\right) \equiv \phi = \left(g_1/a\right)^{\theta} + \beta\left(Ra'/a\right)^{\theta}$$

Setting $a = 1$ in (2.25) and solving out for g_1 and g_2 using (2.25), a system of three equations with three variables $(\phi, a'/a, \kappa)$ is obtained. Call Γ the set of all such trios with κ in the extended reals.

Let $\Gamma^* = \left\{(\phi, a', \kappa) \in \Gamma \text{ such that } \phi \leq \bar{\phi}\right\}$. The solution to our program is the maximum $\phi \in \Gamma^*$. Note that $(0, 0, \infty) \in \Gamma^*$, so Γ^* is non-empty. Obtaining comparative statics on ϕ has proved hard. Next section presents a simple numerical algorithm to compute the values of ϕ , a'/a, and g_1/g_2.

2.6.3 The Value of ϕ and Sensitivity to Parameters

This section presents a general and simple numerical algorithm to compute the values of ϕ and the ratio of savings a'/a for any set of parameter values (β and θ).

Define the following operator, $T\left(\hat{\phi}\right)$:

$$T\left(\hat{\phi}\right) = \max_{g_1, g_2, a'} \left\{g_1^{\theta} + \beta\left(Ra'\right)^{\theta}\right\}$$

subject to :

$$\begin{aligned} g_2^{\theta} + \hat{\phi}\beta\left(a'\right)^{\theta} &\geq R^{\theta} \\ R - g_1 - g_2 &= a' \\ a' &\geq 0 \end{aligned}$$

The operator T is monotone. The parameter ϕ is the maximum fixed point of T such that $\hat{\phi} \in [0, \bar{\phi}]$.

Remark: *The computational approach is as follows : start from some* $\phi_0 \geq \bar{\phi}$ *such that* $T(\phi_0) < \phi_0$. *By monotonicity of* T, *the sequence of positives values* $\phi_{t+1} = T(\phi_t)$ *is monotonically decreasing. It converges to some* $\phi_\infty = \lim_{n\to\infty} T^n(\phi_0)$. *If* $\phi_\infty \in \left[0, \bar{\phi}\right]$

then $\phi = \phi_\infty$.

Notice that $0 = T(0)$. When $\hat{\phi} = 0$, then $g_2 = R$, $g_1 = 0$ and $TV(0) = 0$. This is the worst equilibrium allocation.

Table 2.1 shows the computed biggest fixed point ϕ for different values of β, with $\theta = 0.4$ and $2^\theta (1-\beta)^{1-\theta} > 1$. As β goes downs, ϕ decreases. Also, as predicted from the previous section, all ratios a'/a are smaller than one. As β decreases, the government is saving less (a'/a) and the ability to share (as measured by g_1/g_2) decreases.

Values of β	0.200	0.230	0.260	0.290	0.320	0.350
ϕ	1.013	1.144	1.253	1.341	1.405	1.460
a'/a	0.546	0.643	0.726	0.792	0.836	0.879
g_1/g_2	0.107	0.135	0.162	0.186	0.203	0.222

Table 2.1: Values of ϕ, $\frac{a'}{a}$ and $\frac{g_1}{g_2}$ for $\theta = 0.4$

Figure 2-2 plots the operator T for the values in the table. We can see the existence of two fixed points (zero and the one showed in table 1).

The reading of figure 2-2 might suggest that ϕ is continuous in β. However, this is not true. Table 2.2 presents similar calculations for $\theta = 0.55$. In this case, ϕ (and the policy functions) appear to be discontinuous. For β low enough, cooperation is not sustainaible, and the only possible payoff is the autarky value.

Values of β	0.400	0.440	0.483	0.484	0.500	0.520	0.540	0.560
ϕ	0	0	0	0.920	1.121	1.230	1.292	1.341
a'/a	0	0	0	0.628	0.736	0.845	0.901	0.947
g_1/g_2	0	0	0	0.122	0.131	0.175	0.205	0.233

Table 2.2: Values of ϕ, $\frac{a'}{a}$ and $\frac{g_1}{g_2}$ for $\theta = 0.55$ and different values of β

To explore the discontinuity, figure 2-3 plots the operator $T(.)$ for the corresponding different parameter values. The graph of T is clearly not concave, and *kisses* the forty five degree line around $\hat{\beta} \cong 0.484$ (for $\theta = 0.55$). For values of β lower than that, the operator T never again touches the forty five degree line for values greater than zero. This non concavity of T generates discrete jumps in ϕ for small changes in parameter values.

This is a surprising result. There are some positive β values for which there is a unique equilibrium, the autarky equilibrium, where parties never cooperate, spend everything

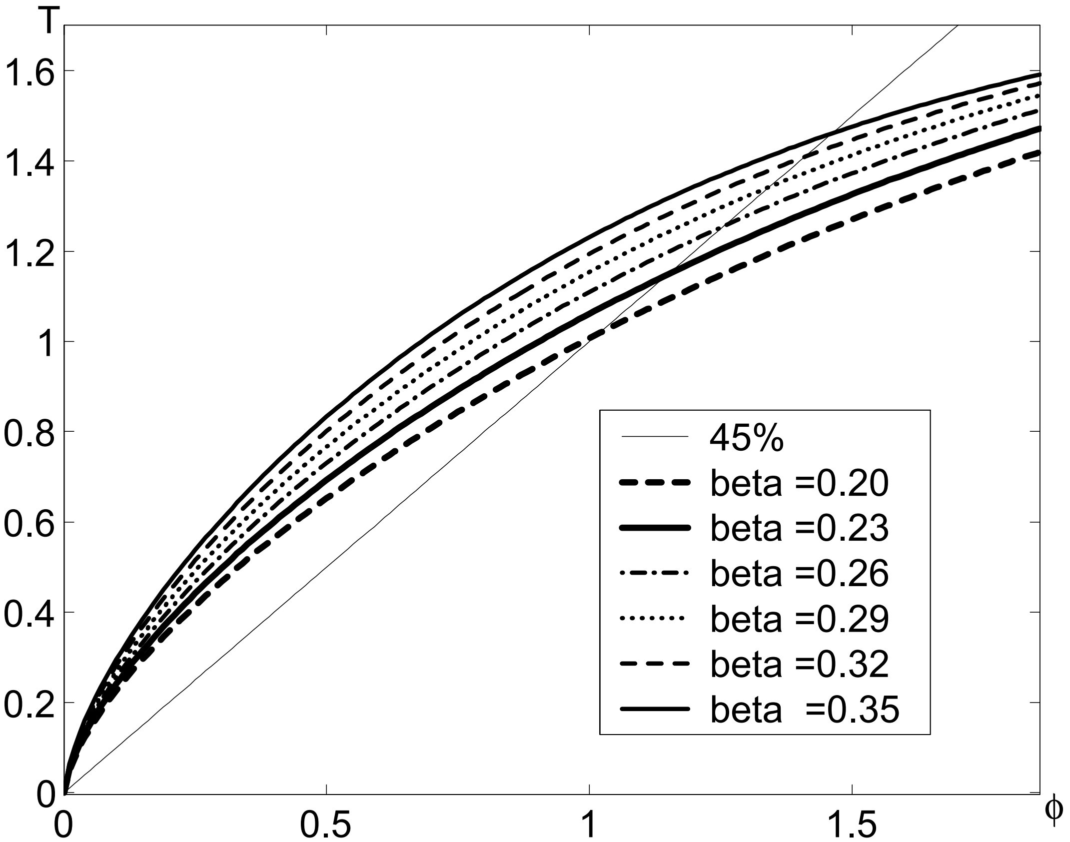

Figure 2-2: The T operator for $\theta = 0.4$ and different values of β

on their own goods, and never save for the future. *Note that this is so even when the parties' future marginal utilities of consumption are equal to infinity.*

Kocherlakota (1993) characterizes the efficient frontier of a game with two players without commitment. He shows that if the marginal utility of consumption in the future under autarky is sufficiently high, there exist always equilibria with more consumption sharing than autarky. The intuition is the following. Every period there is an amount y of income that the parties could share. The parties cannot borrow nor save, and assume for simplicity a political structure identical to the one described in Section 1. Suppose that today the parties are in autarky. Suppose now that they promise each other small amounts of consumption (the party in power gives a small amount of consumption to the party not in power every period). Given that the marginal utility of consumption next period for the party in power under autarky is infinity (she is not receiving any

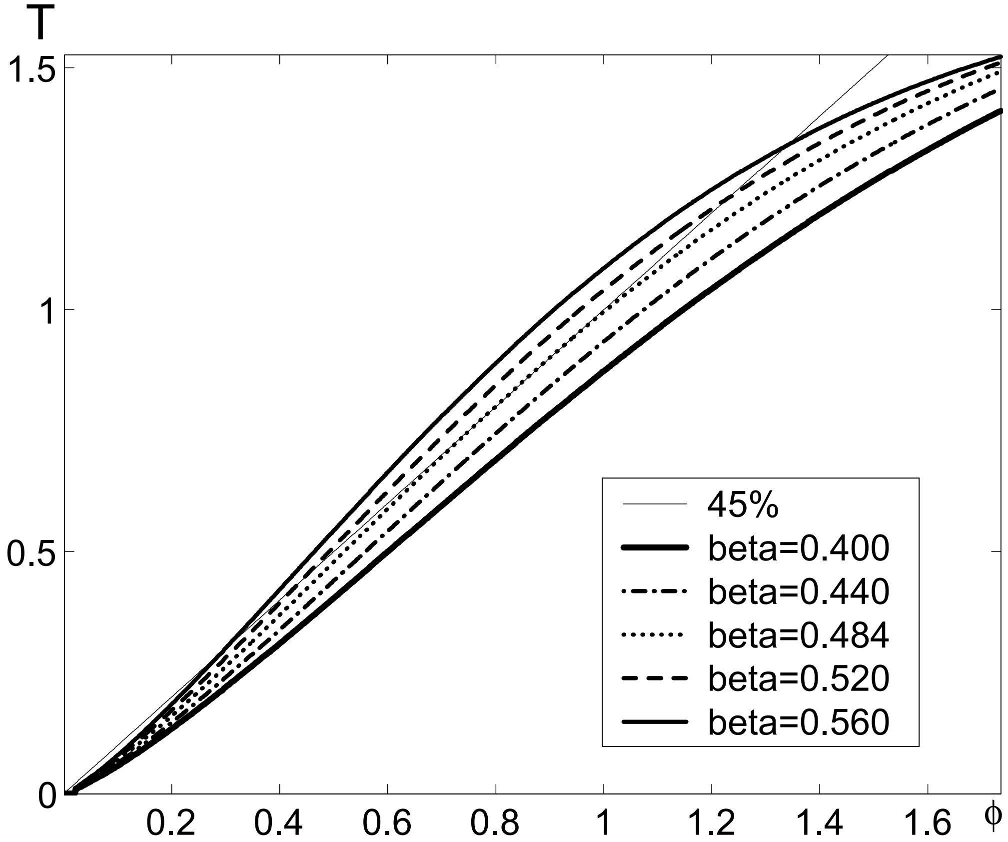

Figure 2-3: The T operator for $\theta = 0.55$ and different values of β

consumption when she is out of power) this strictly increases the utility of both parties above autarky, and hence is incentive compatible.

In the model with savings and no endowment this intuition does not apply. In particular, in the autarky equilibrium, the party tomorrow has no income (no assets have been passed on to the future) and hence nothing to share. The only way to consume in the future is through savings done today by the party in power. In a sharing allocation, the party in power has to provide consumption to the other party and save for the future. She will not receive all the return on those savings however, because the party in power tomorrow will have to be "convinced" to share as well, and hence will have to be provided with consumption. This reduces the incentive to cooperate today, and for low betas and a sufficiently high elasticity of intertemporal substitution, there is no equilibrium but autarky.

Next section explores how the equilibria set might be improved with the use of illiquid assets.

2.6.4 The Role of Illiquid Assets and Efficiency

Suppose now that the government had access to a savings technology that is illiquid[3]. The budget constraint of the government is now

$$\begin{aligned} R\left(k+a\right) &= k' + a' + g_1 + g_2 \\ g_1 + g_2 &\leq Ra \\ a', k' &\geq 0 \end{aligned}$$

Where k denotes the amount saved in the previous period into this illiquid asset. The important thing to notice is that the government can not liquidate s in the current period, so total government spending is constrained to be smaller than the amount of liquid funds, Ra. The government can transform illiquid funds into liquid funds that could be consumed next period.

Because the illiquid funds are not available this period, the current deviation by the party in power can only attain a maximum consumption of Ra. After a deviation has occurred the worst subgame perfect equilibrium is just the consumption of all the income available to the government every period. Given that the use of the illiquid assets is not going to benefit the current party (because she expects the party in the future to spend all the income in her own good), in the worst equilibrium, she won't liquidate any of it[4]. The worst equilibrium payoff to a party in power is then $u(Ra)$, or the consumption of liquid funds.

When are first best allocations incentive compatible with illiquid assets?

A first best allocation is as before characterized by a fraction α allocated to a party out of a constant total spending. For a given total assets $k + a$, there is an incentive compatible first best allocation if for $\alpha = 1/2$ the following holds

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[3] See Laibson(1997).

[4] This could be relaxed without affecting the results.

$$\frac{u\left(\frac{(R-1)(a+k)}{2}\right)}{1-\beta} \geq u(Ra) \tag{2.26}$$

Imposing the condition that total consumption $(R-1)(a+k)$ is done out of the liquid savings:

$$(R-1)(a+k) \leq Ra$$
$$(R-1)k \leq a$$

To minimize the right hand side of (2.26), a is set to its smallest possible value $(R-1)k$. Plugging this back into (2.26), the following is obtained,

Result 2.4 *In the no endowment case with illiquid asset holdings, a first best allocation is incentive compatible if*

$$1 \geq 2^{\theta}(1-\beta)$$

This is a weaker condition than (2.12). In particular,

Result 2.5 *If $\beta > \frac{1}{2}$, an incentive compatible first best allocation always exists.*

The ability to save in illiquid assets, increases the possibility of sustaining a first best allocation. Hence there is a demand for illiquidity as in the case of hyperbolic consumers studied by Laibson (1997). The intuition for this result is very similar, illiquid assets reduce the temptation of the parties to refuse to cooperate, because they can not liquidate easily the wealth of the government. This lowers the incentive constraint for the party in power and more efficient allocations are now sustainable.

2.7 Conclusion

In this paper I studied the behavior of savings under political compromise. Parties in power face a trade-off between consuming today all of the income available or maintaining a reputation for cooperation. I proposed a numerical implementation to calculate the set of efficient subgame perfect equilibria following the work of Abreu, Pearce and Stachetti. It is shown that the savings are inefficiently done in equilibrium, and that parties might demand illiquid assets to improve the equilibrium allocation. It is also shown that the efficient set of subgame perfect equilibria is discontinuous in the parameters, and that there are cases when the marginal utilities of consumption in the future are equal to infinity for all players, and there is still no cooperation.

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