

Investment Cycles and Sovereign Debt Overhang*

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Abstract

We characterize optimal taxation of foreign capital and optimal sovereign debt policy in a small open economy where the government cannot commit to policy, seeks to insure a risk averse domestic constituency, and is more impatient than the market. Optimal policy generates long-run cycles in both sovereign debt and foreign direct investment in an environment in which the first best capital stock is a constant. The expected tax on capital endogenously varies with the state of the economy and investment is distorted by more in recessions than in booms, amplifying the effect of shocks. The government's lack of commitment induces a negative correlation between investment and the stock of government debt, a "debt overhang" effect. Debt relief is never Pareto improving and cannot affect the long-run level of investment. Furthermore, restricting the government to a balanced budget can eliminate the cyclical distortion of investment.

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1 Introduction

This paper explores the joint dynamics of sovereign debt and foreign direct investment in a small open economy. Our analysis brings to the forefront two important political economy considerations. We follow the seminal work of [Thomas and Worrall \(1994\)](#) in that the government cannot commit, leaving capital and debt exposed to expropriation or repudiation. However, in [Thomas and Worrall](#) (and more generally in [Ray, 2002](#)), the government eventually accumulates sufficient assets to overcome its lack of commitment. To this environment we introduce a second prominent political economy implication. Namely, that the risk of losing office makes the government impatient relative to the market. This simple but empirically relevant change in environment leads to dramatically different long run properties of the economy.

We show that the combination of the government's impatience and inability to commit generates perpetual cycles in both sovereign debt and foreign direct investment in an environment in which the first best capital stock is constant. The expected tax on capital endogenously varies with the state of the economy and investment is distorted by more in recessions than in booms, amplifying the effect of shocks. The predictions of the model are consistent with two important phenomena in less developed markets. One is the well known "debt overhang effect" on investment, where current levels of debt negatively effect future investment. Second is the rise in expropriation risk during crises in emerging markets and the depressed level of investment following these crises. We also use our framework to analyze the effect of budgetary restrictions, recently being considered in countries such as Chile and Brazil, on the volatility of consumption and investment.

The model has three types of agents. There are risk averse domestic agents who provide labor inelastically, lack access to financial markets, and do not own capital. There are risk neutral foreigners who invest capital that is immobile for one period and has an opportunity cost given by the world interest rate. Lastly, there is the government that implements fiscal policy on behalf of domestic agents (or a preferred sub-set of agents). Uncertainty is driven by an *i.i.d.* stochastic productivity process. The shock can be interpreted as a productivity shock or a terms of trade shock. This generates a risk that the domestic agents cannot insure. The government provides insurance to domestic agents by taxing or subsidizing foreign capitalists and trading a non-contingent bond with international financial markets. Our choice of an *i.i.d.* process for uncertainty implies that the expected marginal product of capital in the next period is independent of the shock's realization today; and thus, the first-best capital stock is acyclical. This environment allows us to isolate the role of fiscal policy in generating investment and debt cycles.

To understand the separate roles of limited commitment and impatience we first consider the case with full commitment. If the government could commit, optimal fiscal policy (the Ramsey solution) does not distort capital in this economy (similar to [Judd, 1985](#) and [Chamley, 1986](#)). The combination of state contingent taxes and the bond is equivalent to the government having access to a complete set of state-contingent assets, as in [Zhu \(1992\)](#), [Judd \(1992\)](#) and [Chari et al. \(1994\)](#). Under commitment, full insurance is achieved while maintaining an expected foreign tax of zero. The government exploits the fact that capital is ex post inelastic and the risk-neutrality of foreign capitalists to transfer capital income across states. The ex ante elasticity of capital provides the necessary incentive to keep average tax payments at zero. Importantly, the result that there is no distortion of capital holds regardless of the government's discount rate as long as the government can commit. This emphasizes the importance of limited commitment in generating the key results.

Next, we consider the more empirically relevant case when the government cannot commit to its promised tax and debt plan. While the sunk nature of capital allows the government to insure domestic agents, it also tempts the government to renege on tax promises ex post. Similarly, a government may wish to default on its outstanding debt obligations. We show that the optimal taxation problem can be written as a constrained optimal contract between a risk-neutral foreigner (who can commit) and the government (who cannot commit). An optimal allocation under limited commitment is sustained by the off-equilibrium prescription that if the government ever deviates on its tax policy or defaults on its debt obligations, foreign investment will drop to zero, and the country will remain in financial autarky thereafter.

An important feature of the optimal allocation under limited commitment is that when the government's participation constraints bind, capital following high income shocks is strictly greater than capital following low shocks, despite the shocks being *i.i.d.*. This cyclical variation in investment arises due to sovereign debt. The strongest temptation to deviate from the optimal plan arises after receiving the highest income shock. An optimal allocation then accommodates such temptation by prescribing higher domestic consumption. However, consumption smoothing implies that it is optimal to increase future domestic consumption as well, a result that is achieved through a reduction in the stock of sovereign debt. A lower stock of debt relaxes subsequent participation constraints, allowing higher investment.

If the government discounts the future at the market rate, the model behaves as in [Thomas and Worrall \(1994\)](#). While investment depends on the realization of output along the transition, the economy monotonically asymptotes to the first best level of capital and there are no cycles in the long-run. However, we are interested in the case when the government discounts the future at a higher rate than the market. There are important political economy reasons, such as the positive probability of losing office, as in [Alesina and Tabellini \(1990\)](#),

that can justify the higher impatience of the government.¹ In this case capital converges to a unique, non-degenerate ergodic distribution whose support lies strictly below the first best. The government's impatience leads it to bring consumption forward, increasing the stock of debt and therefore reducing the sustainable level of capital in the future. This is why capital lies below the first best level in the long run. At the same time, impatience prevents the accumulation of enough assets to sustain complete risk sharing, leaving consumption in the long run sensitive to shock realizations. Movements in consumption combined with the incentive to smooth consumption inter-temporally generate corresponding fluctuations in the stock of debt carried forward. As the level of debt determines the sustainable level of investment, fluctuations in debt generate corresponding movements in capital. This is why capital is not constant in the long run.

To clarify the role of the government's desire to *insure* domestic agents, we analyze the role that risk aversion plays in generating long run fluctuations. We show that, depending on the dispersion of the underlying shock process, the economy with an impatient government may converge to a degenerate ergodic distribution if agents are risk neutral. In this case, the economy does not fluctuate in the steady state, but instead converges to a constant capital stock less than the first best level. If agents are risk neutral, there is no gain to inter-temporal smoothing, but there is a loss from capital volatility due to concavity in the production function. Efficiency therefore requires that shocks to output are accommodated through changes to current consumption, rather than changes in debt positions which influence investment levels. Risk neutrality therefore breaks the inter-temporal link between shocks today and capital tomorrow stemming from fluctuations in sovereign debt.

We also explore the role of access to sovereign debt markets in generating investment cycles. We analyze the situation in which the government is forced to run a balanced budget, and therefore cannot transfer resources across periods. In this case, distortions to investment will be independent of the current shock in an *i.i.d.* environment. Investment may be distorted, but will be constant. Further, for a discount factor lower than the market rate, it can be the case that under a balanced budget rule investment is first best and consumption is constant. Hence, the government's access to debt markets can increase the volatility of consumption and the distortion of investment.

A recent quantitative literature has emerged on sovereign debt based on the model of [Eaton and Gersovitz \(1982\)](#), beginning with the papers of [Aguiar and Gopinath \(2006\)](#) and [Arellano \(forthcoming\)](#). These models abstract from political economy issues as well

¹[Eaton and Kletzer \(2000\)](#) present an analysis of differential impatience in a sovereign debt model without investment. See also [Krueger and Perri \(2005\)](#) for how a market interest rate below the inverse of the discount factor arises in the general equilibrium of a risk sharing model with lack of commitment.

as investment. This paper contributes to this literature by highlighting the important role that government impatience plays in the joint dynamics of sovereign debt and foreign direct investment.

The interaction of sovereign debt and investment in the model is consistent with the well known debt overhang effect on investment in less developed countries. This negative effect of accumulated debt on investment has been widely explored starting with the work of [Sachs \(1989\)](#) and [Krugman \(1988\)](#). In these papers, the level of debt is assumed to be exogenous, and debt relief is shown to enhance investment and in some cases to generate a Pareto improvement. Differently, in our model, such cyclical debt overhang arises endogenously due to the limited ability of the government to commit. However, at all times the optimal allocation generates payoffs on the Pareto frontier under the limited commitment restriction, and hence debt relief, while benefiting the government, can never generate a Pareto improvement. Furthermore, the existence of a unique long run distribution implies that debt relief programs will at most have short-lived effects. The distortions in investment arise from the lack of commitment of the government and impatience, which are issues that cannot be resolved through debt relief.

The dynamics of the model are also reminiscent of emerging market crises. As predicted by the model, governments often allow foreign capital to earn large returns in booms but confiscate capital income during crises. Moreover, as documented by [Calvo et al. \(2005\)](#), investment remains persistently depressed following a crisis. The most recent crisis in Argentina in January 2002 is a dramatic illustration of this phenomenon. Measures of expropriation risk for Argentina as calculated by the Heritage Foundation and Fraser Institute deteriorated sharply. A similar deterioration of property rights is observed in other emerging market crises, often precipitated by a terms of trade shock or other exogenous drop in income. Our paper rationalizes such behavior. Several recent studies have documented the significant pro-cyclicality of fiscal policy in emerging markets (see [Gavin and Perotti, 1997](#); [Kaminsky et al., 2004](#); and [Talvi and Vegh, 2004](#)). While quarter-to-quarter fluctuations in fiscal policy are interesting, we feel our model is particularly relevant for the interaction of sovereign debt and foreign investment observed at the lower frequency of large crisis episodes.

The paper is organized as follows. Section 2 describes the model environment; Section 3 characterizes the optimal policy under full commitment; Section 4 characterizes the optimal policy under limited commitment; Section 5 restricts the government to a balanced budget and explores the role of risk aversion; and Section 6 concludes. The Appendix contains all proofs.

2 Environment

Time is discrete and runs to infinity. The economy is composed of a government and two types of agents: domestic agents and foreign capitalists. Domestic agents (or “workers”) are risk averse and supply inelastically l units of labor every period for a wage w . Variables will be expressed in per capita units.

The economy receives a shock z every period. One can interpret the shock as a terms of trade shock to a developing country’s exports or a productivity shock. The assumptions underlying the shock process are described below.

Assumption 1. *The shock z follows an i.i.d. process and the realizations of z lie in a finite set $Z \subset \Re$. Let the highest element of Z be \bar{z} and the lowest element be \underline{z} .*

Let $\pi(z)$ denote the associated probability of state z . Let $z^t = \{z_0, z_1, \dots, z_t\}$ be a history of shocks up to time t . Denote by $\pi(z^t)$ the probability that z^t occurs.

Workers enjoy period utility over consumption in history z^t given by $U(c(z^t))$, where U is a standard utility function defined over non-negative consumption satisfying Inada conditions with $U' > 0, U'' < 0$. Let $U_{min} \equiv U(0)$. We make the following assumption about the government’s objective function:

Assumption 2 (Government’s Objective). *The government’s objective function is to maximize the present discounted utility of the workers, discounted at the rate $\beta \in (0, 1)$:*

$$\sum_{t=0}^{\infty} \sum_{z^t} \pi(z^t) \beta^t U(c(z^t)). \quad (1)$$

The government’s discount factor β plays an important role in the analysis. We discuss the determinants of the government’s rate of time preference in detail at the end of the section.

We should note that it is not crucial that the government cares equally about all domestic agents. We could assume that the government maximizes the utility of a subset of agents, such as political insiders or public employees. The analysis will make clear that our results extend to these alternative objective functions as long as the favored agents are risk averse and lack access to capital markets.

Workers provide l units of labor inelastically each period. Moreover, workers do not have access to financial markets. Their consumption is given by:

$$c(z^t) = w(z^t) l + T(z^t), \quad (2)$$

where $T(z^t)$ are transfers received from the government at history z^t and $w(z^t)$ is the competitive wage at history z^t .

As we will see below, we allow the government to borrow and lend from foreigners on behalf of workers. If the government implements the workers' optimal plan, workers and the government can be considered a single entity. The expositional advantage of separating workers from the government is that in practice it is the government that can tax capital and not individual workers. Moreover, the government may not implement the workers' optimal plan. Rather than exclude workers from asset markets entirely, an alternative assumption would be that the government can observe private savings and has a rich enough set of policy instruments to implement a consumption plan for workers, as in [Kehoe and Perri \(2004\)](#). In either case, the important decision problem is that of the government's, which is the focus of the analysis below.

There exists a continuum of risk-neutral foreign capitalists who supply capital, but no labor. The foreign capitalists own competitive domestic firms that produce by hiring domestic labor and using foreign capital. This last assumption is critical: foreign capital is essential for production in the foreign owned sector. The production function of the foreign owned domestic firms is of the standard neoclassical form:

$$y = A(z)f(k, l),$$

where f is constant returns to scale with $f_k > 0$, $f_{kk} < 0$ and satisfying Inada conditions; and A is a positive function.

The capitalists have access to financial markets. We assume a small open economy where the capitalists face the exogenous world interest rate of r . Capital is installed before the shock and tax rate are realized and cannot be moved until the end of the period. We denote by $k(z^{t-1})$ the capital installed at the end of period $t-1$ to be used at time t . The depreciation rate is δ . Capital profits (gross of depreciation) of the representative firm are denoted $\Pi(z^t)$, where

$$\Pi(z^t) = A(z_t)f(k(z^{t-1}), l) - w(z^t)l.$$

The government receives an endowment income each period $g(z)$. This captures, for example, returns to a natural resource endowment sold on the world market. To reflect that income is not zero absent foreign investment, we assume that $g(z) > 0$. The government also taxes capital profits at a linear rate $\tau(z^t)$ and transfers the proceeds to the workers $T(z^t)$. For the benchmark model, we assume the government can trade a non-contingent bond with the international financial markets. Let $b(z^t)$ denote the outstanding debt of the government

borrowed at history z^t and due the next period (which is constant across potential shocks realized at $t + 1$). The government's budget constraint is:

$$g(z^t) + \tau(z^t) \Pi(z^t) + b(z^t) = T(z^t) + (1 + r)b(z^{t-1}). \quad (3)$$

Taking as given a tax rate plan $\tau(z^t)$, firms maximize after-tax profits net of depreciation and discounted at the world interest rate,

$$\mathbb{E}_0 \left[\sum_t \left(\frac{1}{1+r} \right)^t \left((1 - \tau(z^t)) \Pi(z^t) - k(z^t) + (1 - \delta)k(z^{t-1}) \right) \right].$$

Profit maximization and labor market clearing imply the following two conditions:

$$w(z^t) = A(z_t) f_l(k(z^{t-1}), l), \quad (4)$$

and

$$r + \delta = \sum_{z_t \in Z} \pi(z_t) (1 - \tau(z^t)) A(z_t) f_k(k(z^{t-1}), l), \quad (5)$$

where f_i denotes the partial derivative of f with respect to $i = k, l$.

According to equation (5), the expected return to capitalists from investing in the domestic economy net of depreciation should equal the world interest rate r . Given the *i.i.d.* assumption regarding the shocks, optimal capital is a constant in a world without taxes. We denote this first best level of capital by k^* , that is

$$\sum_{z \in Z} \pi(z) A(z) f_k(k^*, l) = r + \delta.$$

Let us define the total output of the economy as $F(z, k, l)$:

$$F(z, k, l) \equiv A(z) f(k, l) + g(z). \quad (6)$$

We are going to impose the following monotonicity assumption that requires that high values of z index high shocks; that is, states when total output is high:

Assumption 3. $F(z, k, l)$ is strictly increasing in z for all $k > 0$ and $l > 0$.

There is a simple way of summarizing the constraints (2), (3) and (4). For this, note that $F_i = f_i$, for $i = k, l$. Equations (2), (3) and (4) can be combined to obtain:

$$\begin{aligned} F(z_t, k(z^{t-1}), l) - (1 - \tau(z^t)) F_k(z_t, k(z^{t-1}), l) k(z^{t-1}) + b(z^t) \\ = c(z^t) + (1 + r)b(z^{t-1}), \end{aligned} \quad (7)$$

where we have used the constant-returns-to-scale assumption (specifically, $f = f_k k + f_l l$) and $F_k = A(z)f_k$ in the derivation. Equation (7) states simply that consumption and debt payments (the right hand side) must equal total output minus equilibrium payments to capital plus new debt.

For the rest of the paper, as labor supply is constant, we will remove the dependence of F on l for simplicity.

The government's discount factor

As already noted, the government's discount factor, β , plays an important role in the subsequent analysis. In [Thomas and Worrall \(1994\)](#), a benevolent government discounts at the world interest rate, that is $\beta = 1/(1+r)$. However, it may be the case that the world interest rate is lower than agents' discount factors, as in the general equilibrium models of [Huggett \(1993\)](#) and [Aiyagari \(1994\)](#). Moreover, there are political economy reasons that may justify a government that discounts the future at a higher rate than its domestic constituency. Perhaps the most direct driver of government impatience is the fact that governments may lose office, as in the canonical model of [Alesina and Tabellini \(1990\)](#). In their model, politicians are impatient because the nature of the political process does not assure the incumbent politicians that they will remain in power in the future. This force for government impatience is prominent in several other political economy models, for example [Grossman and Van Huyck \(1988\)](#) and [Amador \(2004\)](#). In general, there are compelling theoretical reasons why the government's discount factor may differ from the world interest rate. Moreover, there is suggestive empirical evidence, as well. For example, political uncertainty is associated with actions consistent with increased impatience, such as lower levels of foreign reserves and an increased reliance on inefficient systems (see [Aizenman and Marion, 2004](#); and [Cukierman et al., 1992](#)).

Given the above discussion, it is important to consider the consequences of government impatience on the model's predictions. In particular, the case of $\beta(1+r) < 1$ is of particular interest in understanding sovereign debt dynamics and the associated pattern of foreign direct investment. We therefore explicitly include this case in our analysis and proceed under the following assumption:

Assumption 4. *The government discount factor β is such that $\beta(1+r) \leq 1$.*

3 Optimal Taxation under Commitment

Before we proceed to the analysis with limited commitment, as a useful comparison we quickly characterize the optimal fiscal policy under commitment. We show that tax policy is not distortionary and that investment will be constant at the first-best level k^* .

Suppose that the government can commit at time 0 to a tax policy $\tau(z^t)$ and debt payments $(1+r)b(z^t)$ for every possible history of shocks z^t . This ‘‘Ramsey’’ plan is announced before the initial capital stock is invested. Given some initial debt, $b(-1)$, the government chooses $c(z^t)$, $b(z^t)$, $k(z^t)$, and $\tau(z^t)$ to maximize its objective, subject to the budget constraints of the domestic agents and the government as well as firm profit maximization, that is equations (5) and (7).

The problem can be simplified once we recognize that the combination of taxes and a bond is equivalent to a complete set of state-contingent assets. In particular,

Lemma 1. *Let v equal (1) evaluated at the optimum with initial debt $b(-1)$. Then*

$$v = \max_{\{c(z^t)\}, \{k(z^t)\}} \sum_{t=0}^{\infty} \sum_{z^t} \pi(z^t) \beta^t U(c(z^t)) \quad (8)$$

subject to,

$$\sum_{t=0}^{\infty} \sum_{z^t} \frac{\pi(z^t)}{(1+r)^{t+1}} \left(F(z_t, k(z^{t-1})) - (r+\delta)k(z^{t-1}) - c(z^t) \right) \geq b(-1) \quad (9)$$

Conversely, any v that solves this problem is a solution to the Ramsey problem.

The complete markets equivalence results from the ability to transfer resources across states within a period using capital taxes (keeping the average tax constant) plus the ability to transfer resources across periods with the riskless bond. The combination is sufficient to transfer resources across any two histories, as also shown in Judd (1992), Zhu (1992), and Chari et al. (1994).

The scheme exploits the fact that capitalists are risk-neutral and that capital is immobile for one period. In the model with redistribution and affine taxation studied by Werning (2007), it is argued that ex-post capital taxes do not in general complete markets, as capital taxes should also replicate contingent transfers among the heterogeneous agents in the economy. In our model, domestic agents are homogeneous, allowing capital taxes to complete markets. Indeed, from the perspective of the government, given that only foreigners hold the capital stock, an ex-post capital levy in a given state and an external asset that pays the country the same amount in the same state are equivalent: they are both state-contingent

instruments that transfer resources from foreigners to domestic agents.

We characterize some key features of the optimal policy under commitment in the following proposition,

Proposition 1. *Under commitment, for all $z_t, z'_t \in Z \times Z$, and for all $z^{t-1} \in Z^{t-1}$, the optimal fiscal policy has the following features: (i) it provides full intra-period insurance to the workers, $c(\{z_t, z^{t-1}\}) = c(\{z'_t, z^{t-1}\})$; (ii) it smooths consumption across periods with discounting, $U'(c(z^{t-1})) = \beta(1+r)U'(c(z_t, z^{t-1}))$; (iii) at the beginning of every period, the expected capital tax payments are zero and therefore capital is always at the first best level; and (iv) the amount of debt issued is independent of the current shock, $b(\{z_t, z^{t-1}\}) = b(\{z'_t, z^{t-1}\})$.*

Results (i) and (ii) are standard outcomes of models with complete markets and full commitment. Consumption is equalized across states of nature. Consumption trends up, down, or is constant over time, depending on whether the rate of time discount is less than, greater than, or equal to the world interest rate, respectively. Result (iii) follows from the fact that capital only enters the budget constraint, so optimality requires maximizing total output and not distorting investment. This zero tax on capital result is well known in the steady state of neoclassical economies, see for example [Chamley \(1986\)](#), and the stochastic version in [Zhu \(1992\)](#). [Chari et al. \(1994\)](#) obtain a similar result in a business cycle model. [Judd \(1985\)](#) also showed the result holds in a model with redistribution. [Werning \(2007\)](#) also obtains a zero long run tax on capital in a model with heterogeneous agents and lump-sum taxation. In our model, the small open economy assumption implies that capital is infinitely elastic ex-ante and therefore that zero taxation of capital is optimal at all dates and not just asymptotically. See [Chari and Kehoe \(1999\)](#) for a related discussion. Result (iv) indicates how consumption smoothing is implemented using taxes and debt. Taxation is used to transfer resources across states, and debt to transfer resources across time. For example, when z is strictly an endowment shock or when there are only two states, the optimal plan calls for counter-cyclical taxes, with capital taxed more in low endowment states compared to high endowment states. Note that this counter-cyclical taxation does not distort investment. Whether z is an endowment or productivity shock, the amount of debt issued however, is independent of the current shock in an *i.i.d.* environment. The resulting fiscal deficit is acyclical. The results in this section show that a government with commitment will not amplify shocks through its tax policy. This holds independent of the relation between β and r – even if the government were impatient relative to the market, there will be no distortion of investment as long as the government can commit.

4 Optimal Taxation with Limited Commitment

Once the investment decision by the capitalists has been made, the government has an incentive to tax capital as much as possible and redistribute the proceeds to the workers. Similarly, a government has an incentive to default on its outstanding debt obligations. Thus, the optimal tax and debt policy under commitment may not be dynamically consistent. In this section we analyze the implications of the government's inability to commit.

To constrain the government's ability to expropriate foreign income to plausible levels, we place an upper bound on the capital income tax rate:

Assumption 5 (A Maximum Tax Rate). *The tax rate on profits cannot be higher than $\bar{\tau} = 1$.*

That is, the most the government can tax in any state is one-hundred percent of profits.

The goal is to characterize efficient equilibria of the game between the capitalists and the government. We make the standard assumption that the external financial markets can commit to deny access in case of a deviation by the government:

Assumption 6. *If the government ever deviates from the prescribed allocation on either taxes or debt payments, the country will remain in financial autarky forever; specifically the government would not be able to issue debt or hold external assets.*

That is, foreign creditors can commit to punish default with complete exclusion, even from savings. This is the harshest penalty that respects sovereignty.

Efficient allocations are implemented with the threat of the worst punishment if the government deviates on *either* taxes or debt payments. As noted above, the worst outcome in financial markets is permanent exclusion. Conditional on financial autarky, the worst equilibrium of the game between foreign owners of capital and the government is zero investment, that is $k_{aut} = 0$. In particular, the government's best response to any positive investment conditional on no future investment and financial autarky is to tax all capital income (i.e. set the tax rate to $\bar{\tau}$). The foreign investors' best response to $\bar{\tau}$ is to invest zero. Therefore, zero investment is always an equilibrium under financial autarky. It is the worst equilibrium as it minimizes the government's tax base at all histories. Let V_{aut} denote the continuation value of the government in autarky. Specifically,

$$V_{aut} = \sum_{z \in Z} \pi(z) \frac{U(F(z, 0))}{1 - \beta}. \quad (10)$$

This assumes that the installed capital cannot be operated (or sold) by the government upon

deviation.² We assume that $\bar{\tau}$ does not bind along the equilibrium path, but places an upper bound on seized income if the government deviates.³

Definition. Given an initial debt $b(-1)$, an **optimal allocation under limited commitment** is a sequence of functions $c(z^t)$, $b(z^t)$, $\tau(z^t)$, and $k(z^t)$ such that (1) is maximized, constraints (2)-(5) hold, and the government at all histories prefers the continuation allocation to deviating and taxing capital at the highest possible rate $\bar{\tau}$, and/or defaulting on its debt obligations:

$$\sum_{i=0}^{\infty} \sum_{z^{t+i}} \pi(z^{t+i}|z^t) \beta^i U(c(z^{t+i})) \geq U(F(z_t, k(z^{t-1}))) + \beta V_{aut}, \forall z^t. \quad (11)$$

Note that the payoff after deviation is independent of the assets held by the government, hence if the government defaults on its tax promises while holding positive assets, $b(z^t) < 0$, it will lose them. It is not relevant for an optimal allocation to specify what happens to the seized assets as long as they are lost to the government. For the rest of the paper, whenever we refer to optimal allocations, we will be referring to optimal allocations under limited commitment.

4.1 A Recursive Formulation

Let us denote by v the maximal amount of utility attainable to the government in an optimal allocation, given that it has issued an amount $b \in \mathbf{b}$ to the foreign financial markets, where \mathbf{b} denotes the set of possible debt levels for which the constraint set is non-empty. We also impose that the set \mathbf{b} is bounded below (that is, assets have a finite upper bound). We discuss below that this bound is not restrictive.

Let us denote by B the function such that $(1+r)b = B(v)$, for any $b \in \mathbf{b}$. We characterize the constrained optimal allocations recursively. Histories are summarized by promised expected discounted utility v for the government. We initially consider all v in a closed interval $[V_{aut}, V_{max}]$, where V_{max} is the value corresponding to the maximal asset level. We assume $V_{max} \geq U(F(\bar{z}, k^*)) + \beta V_{aut}$. The right hand side of this inequality represents the utility obtained by deviation if capital is at its first best level and z is its maximum realization. The lower bound V_{aut} follows immediately from the government's lack of commitment. Once

²Allowing the government to sell off or immediately consume a part of the capital stock after deviation will not change the problem in a significant manner, as long as the outside option remains increasing in the current shock.

³Implicitly, the previous section assumed that $\bar{\tau}$ is greater than the maximal tax rate under the Ramsey plan (that is, the Ramsey plan is feasible without imposing negative post-tax profits on capital).

we have defined $B(v)$ on $[V_{aut}, V_{max}]$, we will characterize and restrict attention to the subset of $[V_{aut}, V_{max}]$ for which $B(v) \in \mathbf{b}$.

We now show that optimal allocations solve the following Bellman equation in which the state variable is v , and the choice variables are the capital stock, state-contingent flow utility (u), and a state-contingent promised utility (ω) which will be next period's state variable. Let Ω define the space of possible choices, where $\omega(z)$ is restricted to $[V_{aut}, V_{max}]$, and $u(z) \geq U_{min}$ and $k \geq 0$. Let the function $c(u)$ denote the consumption required to deliver utility u (i.e., $U(c(u)) = u$). We can say that,

Proposition 2. *Let $(c_0(z^t), k_0(z^t))$ represent an optimal allocation given initial debt $b(-1) \in \mathbf{b}$. Let v represent the government's utility under this allocation. Then $(1+r)b(-1) = B(v)$, where $B(v)$ is the unique solution to the following recursive problem:*

$$B(v) = \max_{(u(z), \omega(z), k) \in \Omega} \sum_{z \in Z} \pi(z) \left[F(z, k) - c(u(z)) - (r + \delta)k + \frac{1}{1+r} B(\omega(z)) \right] \quad (12)$$

subject to

$$v \leq \sum_{z \in Z} \pi(z) [u(z) + \beta \omega(z)] \quad (13)$$

$$U(F(z, k)) + \beta V_{aut} \leq u(z) + \beta \omega(z), \quad \forall z' \in Z. \quad (14)$$

Moreover, the sequence $(c_0(z^t), k_0(z^t))$ satisfies the recursive problem's policy functions (iterating from the initial v).

Conversely, let $(c_1(z^t), k_1(z^t))$ be a sequence generated by iterating the recursive problem's policy functions starting from an initial v for each shock history z^t and $B(v) \in \mathbf{b}$. Then, $(c_1(z^t), k_1(z^t))$ is an optimal allocation starting from an initial debt $(1+r)b(-1) = B(v)$.

The first constraint (13) is the promise keeping constraint that ensures the government enjoys (at least) the promised utility v . The second constraint (14) is the participation constraint. This ensures that the government never has the incentive to deviate along the equilibrium path. As discussed above, consumption during deviation is productive output plus the endowment. The continuation value post deviation is V_{aut} defined in (10). Note that constraint (13) can be treated as an inequality because more utility can be offered to the country without violating previous participation constraints.

The proposition tells us that an optimal allocation sits on the (constrained) Pareto frontier defined by the government's welfare and the bond holders' welfare (subject to the requirement that capital is always paid its opportunity cost). Note that the objective in the recursive problem represents payments to the bond holders.

Our problem hence collapses to the problem of finding a constrained efficient contract between a representative risk-neutral foreigner (who can commit) and the government (who cannot commit). We can thus map our problem to those studied by [Thomas and Worrall \(1994\)](#) and [Albuquerque and Hopenhayn \(2004\)](#). Differently from these papers, we carry out our analysis for concave utility functions and for general relative discount factors.⁴ As we will show below, these two elements have dramatic implications for the behavior of the economy in the long run.

The operator defined by (12) maps bounded functions into bounded functions. It also satisfies Blackwell's sufficient conditions for a contraction. The operator maps the set of continuous, non-increasing functions into itself. Standard arguments therefore imply that,

Lemma 2. *The value function $B(v)$ is non-increasing and continuous.*

The participation constraints (14) are not necessarily convex due to the presence of $U(F(z, k))$ on the left hand side of the inequality. So, to proceed with our characterization of the optimum we make the following change in variable, and define $h \equiv \mathbb{E}A(z)f(k) - (r + \delta)k$ for $k \in [0, k^*]$, and denote the optimal h as $h^* = \mathbb{E}A(z)f(k^*) - (r + \delta)k^*$. Note that h is strictly monotonic on $[0, k^*]$. Let $K(h)$ denote the inverse mapping from $[0, h^*]$ to $[0, k^*]$, such that $k = K(h)$.

The following assumption will ensure the convexity of the problem:

Assumption 7. *For all $z \in Z$, $U(F(z, K(h)))$ is convex in h for $h \in [0, h^*]$.*

By substituting h for k as the choice variable, this assumption ensures that the participation constraints are convex in h , while maintaining the concavity of the objective function (as long as B is concave). Note that the condition must hold state by state.⁵

The condition required by Assumption 7 can be written as

$$\frac{A(z)f_k^2(k)U''(F(z, k))}{f_{kk}(k)U'(F(z, k))} \leq \frac{r + \delta}{\mathbb{E}A(\tilde{z})f_k(k) - (r + \delta)}, \quad (15)$$

for all $k < k^*$. Heuristically, this condition limits the concavity of the utility function relative to the concavity of $f(k)$. For example, if the utility function is linear, the left hand side goes to zero. Note also that the condition is always satisfied in a neighborhood of k^* . As an example, consider the standard utility function and a production function $U(c) = c^{1-\sigma}/(1 - \sigma)$ and

⁴Although [Thomas and Worrall \(1994\)](#) study a concave utility case in the appendix, they only analyzed the case where $\beta(1 + r) = 1$.

⁵An alternative condition that only needed to hold in expectation was used in the working paper version ([Aguiar et al., 2007](#)). However, Assumption 7, for which we thank a referee, simplifies the proof of concavity considerably.

$F(z, k) = zk^\alpha + g_0z$. Then, the condition is satisfied if $\alpha \in (1/2, 1)$ and g_0 is sufficiently large or σ is sufficiently close to zero.⁶

In the appendix we prove that Assumption 7 implies concavity of the foreigner's value function and that optimal policies are interior:

Proposition 3. *Under the stated assumptions, (i) the value function $B(v)$ is concave and differentiable on $[V_{aut}, V_{max}]$; (ii) There exists $V_{min} > V_{aut}$, such that $B'(v) = 0$ for all $v \in [V_{aut}, V_{min}]$, promise keeping holds with strict equality for $v \geq V_{min}$, and $\mathbf{b} = [B(V_{min}), B(V_{max})]$; (iii) $B(v)$ is strictly decreasing for $v \in (V_{min}, V_{max}]$ and strictly concave for $v \in [V_{min}, V_{max}]$; and (iv) for each $v \in [V_{aut}, V_{max}]$, there exists an optimal $(k, u(z), \omega(z))$ with $k > 0$, and such that there exists non-negative multipliers $(\gamma, \lambda(z))$ that satisfy*

$$c'(u(z)) = \gamma + \frac{\lambda(z)}{\pi(z)} \quad (16)$$

$$B'(\omega(z)) = -\beta(1+r) \left(\gamma + \frac{\lambda(z)}{\pi(z)} \right) \quad (17)$$

$$\sum_z \pi(z) F_k(z, k) - (r + \delta) = \sum_z \lambda(z) U'(F(z, k)) F_k(z, k), \quad (18)$$

with $-B'(v) = \gamma$.

Statement (i) of the proposition is that the value function is concave, which requires Assumption 7. Statement (ii) is that the promise keeping constraint does not bind in the neighborhood of V_{aut} . That is, an optimal allocation does not deliver utility below some threshold $V_{min} > V_{aut}$. This places a lower bound on v , which corresponds to the upper bound on b inherent in \mathbf{b} . There is no optimal allocation that delivers a debt level greater than $B(V_{min})$. Statement (iii) strengthens statement (i) in that $B(v)$ is strictly concave over the relevant region. Statement (iv) indicates that there is an interior solution for each v . The function $B(v)$ is depicted in Figure 1.

Let $g^i(v)$ denote the policies in an optimal plan for $i = u(z), \omega(z)$, and k , at state v . We can immediately derive a number of properties of the optimal plan.

Proposition 4. *In an optimal allocation,*

(i) $g^i(v)$ is single-valued and continuous for $i = u(z), \omega(z)$, and k , for all $v \in [V_{min}, V_{max}]$;

⁶Specifically, that is:

$$g_0 > \left(\frac{r + \delta}{\alpha E z} \right)^{\frac{\alpha}{\alpha-1}} \frac{1 + \alpha(\sigma - 1)}{2\alpha - 1} \min \left\{ \left(\frac{\sigma(2\alpha - 1)}{\alpha(\sigma - 1) + 1} \right)^{\frac{\alpha}{1-\alpha}}, 1 \right\}$$

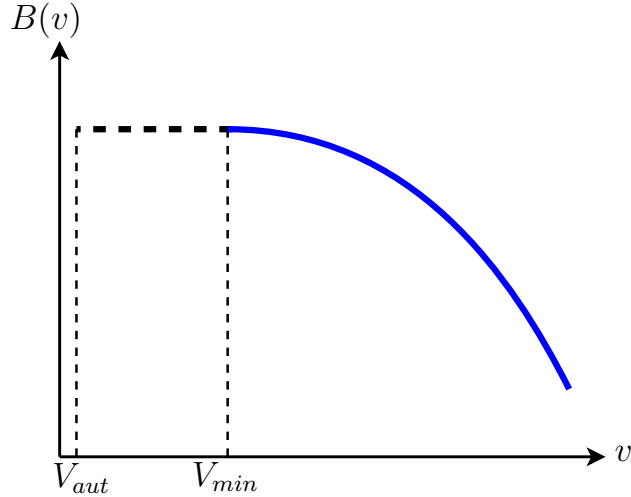


Figure 1: This figure depicts the function $B(v)$. The solid portion associated with $v \geq V_{min}$ is the Pareto frontier.

- (ii) For all $v \in [V_{aut}, V_{max}]$, $g^k(v) \leq k^*$;
- (iii) For any $v \in [V_{aut}, V_{max}]$, if the participation constraints are slack for a subset $Z_o \subset Z$, then $c(g^{u(z)}(v))$ is constant for all $z \in Z_o$. Moreover, $B'(g^{\omega(z)}(v)) = \beta(1+r)B'(v)$ for all $z \in Z_o$;
- (iv) If for some $v \in [V_{aut}, V_{max}]$ and $(z', z'') \in Z \times Z$ we have that $g^{u(z')}(v) \neq g^{u(z'')}(v)$ or $B'(g^{\omega(z')}(v)) \neq \beta(1+r)B'(v)$, then $g^k(v) < k^*$;
- (v) For any $v \in [V_{aut}, V_{min}]$, $g^{\omega(z)}(v) \geq V_{min}$ with a strict inequality for at least one $z \in Z$, and for $v \in (V_{min}, V_{max}]$, $g^{\omega(z)}(v) > V_{min}$ for all $z \in Z$.

Part (i) of the proposition states that policies are unique and continuous, which follows directly from the strict concavity of the objective function. Part (ii) states that capital never exceeds the first-best level. This can be seen from (18) and the fact that multipliers are non-negative.⁷ Part (iii) states that the planner will always implement insurance across states and across time to the extent possible. If two states have unequal consumption and slack constraints, it is a strict improvement (due to risk aversion) to narrow the gap in consumption. Part (iv) of the proposition states that if the government fails to achieve perfect insurance, it will also distort capital. To see the intuition for this result, suppose that capital was at its first-best level but consumption was not equalized across states. Then,

⁷Benhabib and Rustichini (1997) show that in a deterministic closed economy model of capital taxation without commitment, there are situations where capital is subsidized in the long run, pushing capital above the first-best level. In our case, with an open economy, such a situation never arises.

some participation constraints must be binding in the states with high consumption. The government could distort capital down slightly to relax the binding participation constraints. This has a second-order effect on total resources in the neighborhood of the first-best capital stock. However, the relaxation of the participation constraints allows the government to improve insurance. Starting from an allocation without perfect insurance, this generates a first-order improvement in welfare. Finally, part (v) states that eventually, an optimal allocation uses only continuation values on the interior of the Pareto frontier.

Define $V^* = U(F(\bar{z}, k^*)) + \beta V_{aut}$. Note that for any $v \geq V^*$ we have that $g^k(v) = k^*$.⁸ And that for $v < V^*$ at least one participation constraint will be binding. The next proposition further characterizes the optimal allocation,

Proposition 5. *In an optimal allocation,*

- (i) $g^k(v)$ is non-decreasing in v , and strictly increasing for all $v \in [V_{min}, V^*]$;
- (ii) $g^{\omega(z)}(v)$ and $g^{u(z)}(v)$ are strictly increasing in v for all $v \in [V_{min}, V_{max}]$;
- (iii) $g^{\omega(z_1)}(v) \geq g^{\omega(z_0)}(v)$ if $z_1 > z_0$, and $g^{\omega(\bar{z})}(v) > g^{\omega(\underline{z})}(v)$ for all $v \in [V_{min}, V^*]$.

Result (i) states that capital is increasing in promised utility or is at the first best. Result (ii) tells us that utility flows and continuation values are increasing in promised utility as well. Result (iii) states that future promised utility is non-decreasing in the realization of the endowment. In other words, realizations of the shock generate a monotone “spreading out” of continuation values. If $v < V^*$, then insurance across states is not perfect and there will be at least one pair of states where continuation values are strictly different.

The reason continuation values are relatively high following a high shock is due to limited commitment and the temptation to deviate. The strongest temptation to deviate from the optimal plan arises after receiving the highest income shock. An optimal contract naturally accommodate such temptation by prescribing higher domestic utility in case of a high income shock today. Consumption smoothing implies that it is optimal to increase future utility flows as well as the current flow utility, a result that is achieved through a higher continuation value.

The spreading out of continuation values and the fact that capital is increasing in promised utility implies the following:

Proposition 6 (Procyclicality). *In an optimal allocation, $k(z_t, z^{t-1}) \leq k(z'_t, z^{t-1})$ for $z_t < z'_t$. Also, if $k(\underline{z}, z^{t-1}) < k^*$ then $k(\underline{z}, z^{t-1}) < k(\bar{z}, z^{t-1})$.*

⁸If $v \geq U(F(\bar{z}, k^*)) + \beta V_{aut}$ then, ignoring the participation constraints, it is optimal to set $u(z) + \beta w(z) = v$ for all z and $k = k^*$, which satisfies the participation constraints.

This proposition states that capital responds to shocks in a way that prolongs their impact, in an environment in which the shocks are *i.i.d.*. A similar result was obtained by [Thomas and Worrall \(1994\)](#), but only along a monotonic transition to a steady state level of capital. As we will show in the next section, for general discount rates, such pro-cyclicality result is maintained at a non-degenerate ergodic distribution.

4.2 Long Run Properties

If the government discounts the utility flows at the world interest rate, from (17) it follows that $-B'(\omega(z)) = -B'(v) + \lambda(z)/\pi(z)$. Given that the multipliers are non-negative and that B is strictly concave on $(V_{min}, V_{max}]$, this implies that v is weakly increasing over time and strictly increasing when the participation constraint binds. If the initial v lies below V^* , then $\lim_{t \rightarrow \infty} v_t = V^*$. The fact that the continuation value policies, $g^{w(z)}(v)$, are strictly increasing in v , implies that $v_t < V^*$ for all t . If the initial v lies above V^* , then no participation constraint ever binds, and v remains constant at its initial value. Monotonicity of v implies that in the long run capital monotonically approaches the first best. This result is a major feature of the models of [Thomas and Worrall \(1994\)](#) and [Albuquerque and Hopenhayn \(2004\)](#). In this environment, eventually enough collateral, in the shape of foreign assets, is built up so that the participation constraint relaxes and the first best capital level obtains. Therefore, the amplification and persistence results of Proposition 6 only hold along the transition, but not in the steady state.

An alternate situation is one in which the government is impatient relative to the world interest rate. In this case, the government has a preference for early consumption. However, bringing consumption forward tightens the participation constraints in the future, distorting investment. In fact, if the government is relatively impatient, promised utilities and capital converge to a non-degenerate ergodic distribution:

Proposition 7 (Impatience). *If $\beta(1+r) < 1$, in an optimal allocation, v and k converge to unique, non-degenerate ergodic distributions with respective supports that lie strictly below V^* and k^* .*

Impatience makes the persistence and cyclicity generated by limited commitment a permanent feature of the economy. And the economy never escapes the range in which capital is distorted (not even asymptotically).

To visualize how the economy converges to the ergodic distribution, we plot the policy functions for continuation utility ($g^{w(z)}(v)$) in figure 2. We assume two states for the endowment shock, \bar{z} and \underline{z} . Proposition 5 states that the policy function for the high shock lies above the policy for the low shock (strictly above for any $v < V^*$), and they are both

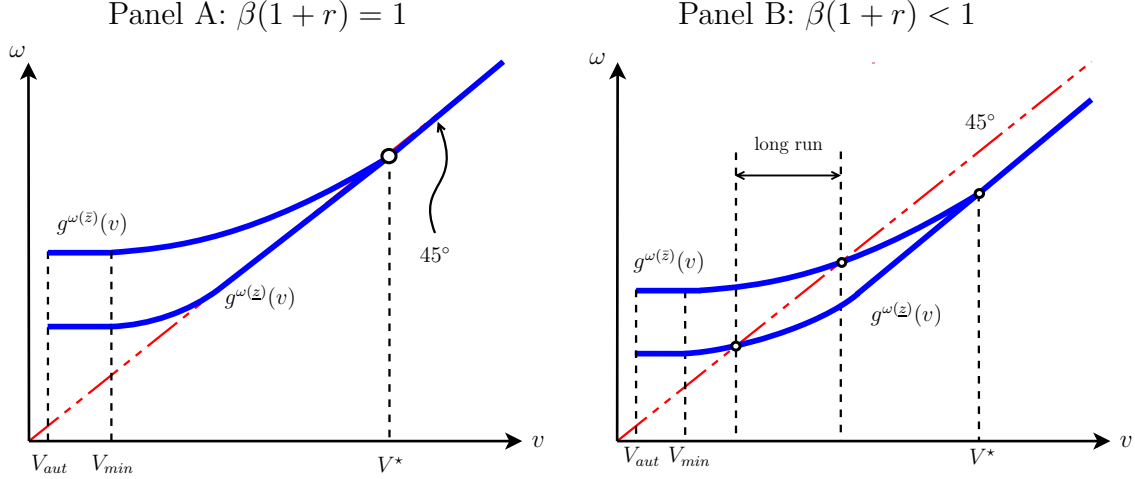


Figure 2: This figure depicts policy functions for next period's promised utility as a function of the current promised utility. The shock takes two possible values. The “top” solid line represents the policy if the shock is high and the “bottom” solid line represents the policy if the endowment shock is low. The dashed ray is the 45 degree line. Panel A represents the case when $\beta(1+r) = 1$ and Panel B represents the case when $\beta(1+r) < 1$.

increasing in v . The policy functions lie strictly above the 45 degree line at V_{aut} because $g^z(v) \geq V_{min} > V_{aut}$ by Proposition 4 part (v).

Panel A assumes $\beta(1+r) = 1$ and panel B assumes $\beta(1+r) < 1$. Panel A indicates that for $v \geq V^*$, the policy functions are equal (insurance across states) and on the 45 degree line (smoothing across periods). For $v < V^*$, the policy functions lie above the 45 degree line. Therefore, starting from some $v < V^*$, the promised utilities increase over time, approaching V^* in the limit. Panel B indicates that when $\beta(1+r) < 1$, for any $v \geq V^*$, the policy functions lie strictly below the 45 degree line. To see this, note that from (17) and the envelope condition,

$$-B'(\omega(z)) = \beta(1+r) \left(-B'(v) + \frac{\lambda(z)}{\pi(z)} \right).$$

When all participation constraints are slack, $\beta(1+r) < 1$ and strict concavity of the value function imply that $\omega(z) < v$. The policy functions intersect the 45 degree line at different points, which indicate the limits of the ergodic distribution. The limiting distribution is non-degenerate from Proposition 5. The uniqueness of the distribution is shown by proving that the minimum value where the policy function $g^{\omega(\bar{z})}(v)$ crosses the 45 degree line is always strictly above the maximum value where the policy function $g^{\omega(\underline{z})}(v)$ crosses the 45 degree line. This means that there exists a middle point, \hat{v} , such that $g^{\omega(\underline{z})}(v) < v$ for $v \geq \hat{v}$,

and $g^{\omega(\bar{z})}(v) > v$ for $v \leq \hat{v}$. Therefore, \hat{v} constitutes a mixing point, and a unique ergodic distribution follows.

4.3 Discussion of the Benchmark Model

Within the limiting distribution, a low shock will lower promised utility and a high shock will raise it, reflecting that the government's utility oscillates within this range. Capital will converge to a corresponding non-degenerate distribution in which $k < k^*$, with capital oscillating one-for-one with promised utility. The level of capital and the level of debt (the inverse of promised utility) of the economy are negatively correlated. The limited commitment of the government therefore generates a debt over-hang effect. Following high shocks the government accumulates assets. This slackens the participation constraint of the government in the future, reducing the incentive to deviate on taxes and therefore supporting higher investment. On the other hand, following low shocks, the government accumulates debt which raises the incentive to deviate and lowers investment. This mechanism differs from that described in [Sachs \(1989\)](#) and [Krugman \(1988\)](#) where a large level of debt reduces domestic investment because debt payments behave like a tax on investment. In the earlier debt over-hang literature, debt relief can not only raise investment but also be Pareto improving. In our environment, debt relief can benefit the government, but it is never a Pareto improvement, since the economy is at all points on the constrained Pareto frontier.

In addition, since the long-run distribution of investment is unique, debt relief programs cannot have a long-run effect on investment. The distortions in investment arise from the lack of commitment of the government and impatience, which are issues that cannot be resolved through debt relief.

The pattern delivered by the model is reminiscent of emerging market crises, as discussed in the introduction. In many instances the increased fear of expropriation during a downturn generates a sharp drop in foreign investment, amplifying the decline in output. Optimal tax policy in the presence of limited commitment is consistent with such empirical regularities.

While the government is providing insurance to the domestic agents, this does not necessarily imply that government expenditures are higher during bad states. The model makes no distinction between private consumption and government provision of goods to the domestic agents. Nevertheless, it is the case that the sum of consumption and government expenditures is positively correlated with the shocks in the model, a fact consistent with the data (see [Kaminsky et al., 2004](#)).

Net foreign liabilities in our model can be defined as debt plus foreign capital. The change in net foreign assets, the current account, is typically counter-cyclical in the data,

particularly for emerging markets (see [Aguiar and Gopinath, 2007](#)). In our model, a positive shock generates an inflow of foreign direct investment inducing a deterioration of the current account. On the other hand, debt declines following a high shock, generating an improvement of the current account. The net effect on the current account is theoretically ambiguous, a standard outcome in a model with transitory shocks.

Finally, note that a common feature of models of insurance with limited commitment is that the participation constraints tend to bind when the endowment is high. This results from the fact that insurance calls for payments during booms and inflows during downturns. However, in precisely an environment that emphasizes insurance, we show that distortions of investment are greater during recessions because of the need to accumulate debt. This in turn makes low taxes in the future more difficult to sustain, depressing investment today.

5 The Importance of Risk Aversion and Debt

In the benchmark environment, the government uses taxes and debt to insure a risk averse domestic constituency. In this section, we explore the relative roles that insurance and debt play in the investment dynamics described above. Specifically, we consider two alternative environments: (i) domestic agents are risk neutral; and (ii) the government runs a balanced budget. We show that absent risk aversion, under certain parameter specifications, the economy converges to a degenerate distribution in the long run. Similarly, when the government is constrained to a balanced budget, there also is no cyclicalty in foreign investment.

5.1 Risk Neutral Domestic Agents

To highlight the role of concavity of the utility function, in this subsection we impose instead the assumption that domestic agents utility flow is given by $U(c) = c$, with the additional restriction that $c \geq 0$.

In this environment, the government has no incentive to use taxes and debt to smooth consumption across states or time. The question we address is whether fiscal policy continues to induce cycles in investment absent the need for insurance. As shown in [Thomas and Worrall \(1994\)](#), if $\beta(1+r) = 1$, the linear case shares the same long run properties as the risk averse case, namely, investment converges to the first best level. The case of interest is then when $\beta(1+r) < 1$.

Let us define \tilde{k} as the value that solves:

$$\mathbb{E}F_k(z, \tilde{k}) = \frac{r + \delta}{\beta(1+r)}. \quad (19)$$

The next proposition states that if agents are risk neutral, there are parameter values such that the economy converges to a degenerate distribution, albeit one with sub-optimal investment \tilde{k} . This is in contrast with what we obtained for the risk averse case when $\beta(1+r) < 1$.

Proposition 8. *Suppose $U(c) = c$, with $c \geq 0$. If $F(\underline{z}, \tilde{k}) \geq \beta \mathbb{E} [F(z, \tilde{k}) - F(z, 0)]$, where \tilde{k} is given by equation (19), then in an optimal allocation the economy converges to a degenerate ergodic distribution with constant capital level \tilde{k} .*

Note that if $\beta(1+r) < 1$, then the long run capital stock is sub-optimal, as $\tilde{k} < k^*$, but constant. The fact that capital is sub-optimal follows from impatience, as in the risk averse case. The government would like to bring consumption forward, and it does so by accumulating debt. The optimal debt level trades off the desire to consume early against the fact that debt depresses investment. However, in the risk neutral case, impatience does not necessarily imply that capital fluctuates indefinitely. To understand why, note that concavity of the production function implies that it is inefficient for investment to vary over time. Movements in investment are driven by movements in debt, given that debt levels influence the government's incentives to expropriate. In the risk averse case, debt was necessary to smooth consumption over time. However, with risk neutral utility, there is no need to smooth consumption over time or across states. Therefore, the optimal plan has a constant debt level and adjusts contemporaneous consumption to absorb all the fluctuations in z . More precisely, contemporaneous consumption absorbs all shocks subject to the constraint that consumption cannot be negative. The condition $F(\underline{z}, \tilde{k}) \geq \beta \mathbb{E} [F(z, \tilde{k}) - F(z, 0)]$ ensures that the shocks are not so dispersed that the optimal allocation is constrained by the non-negativity of consumption.

When shocks are additive, this condition is always satisfied:

Corollary 1 (Endowment Shocks). *Suppose that $F(z, k) = f(k) + g(z)$, that is $A(z) = 1$ for all z , then capital converges to a degenerate ergodic distribution at \tilde{k} .*

Under risk neutrality, debt is not used to smooth consumption inter-temporally. However, debt does allow consumption to be brought forward, reducing the level of investment sustainable in the steady state. That is, as in the risk averse environment, there is a trade off between early consumption and building up enough foreign assets to sustain high investment. In the risk neutral case, we can solve explicitly for the level of debt in the degenerate steady state:

Lemma 3. *Suppose that $\beta(1+r) < 1$ and $F(\underline{z}, \tilde{k}) \geq \beta \mathbb{E} [F(z, \tilde{k}) - F(z, 0)]$, with \tilde{k} defined in equation (19). Then \tilde{b} , the amount of debt outstanding in the steady state, is given by*

$$r\tilde{b} = \beta \left(\mathbb{E}[F(z, \tilde{k}) - F(z, 0)] - (1+r)\mathbb{E}F_k(z, \tilde{k})\tilde{k} \right)$$

Note that the sign of outstanding debt is ambiguous, that is, the government may be a net debtor or creditor. In particular, \tilde{b} is positive if $\mathbb{E}[F(z, \tilde{k}) - F(z, 0) - F_k(z, \tilde{k})\tilde{k}] > r\mathbb{E}F_k(z, \tilde{k})\tilde{k}$. Given that \tilde{k} is bounded away from zero, the term on the left is strictly positive due to the concavity of the production function. Therefore, the country will be a long-run debtor if r is sufficiently small. For example if $f(k) = k^\alpha$, then the condition in Lemma 3 implies that the country is a debtor in the degenerate distribution if and only if $\alpha > 1/(1+r)$. It may seem surprising in this example that whether the country is a debtor or creditor does not depend on β . However, the absolute magnitude of \tilde{b} as well as the level of capital \tilde{k} depend on β (see equation 19). More generally, whether the country is a long run debtor or creditor depends on the parameters of the problem. However, as long as $\beta(1+r) < 1$, even if the government holds positive foreign assets, these assets will not be sufficient to sustain first best investment.

In general however, the distribution of shocks may be such that the condition for Proposition 8 fails and it is infeasible to implement the constant investment allocation. We now state a partial converse to Proposition 8 for the linear case with $\beta(1+r) < 1$, where we continue to use V_{min} to denote $\max\{v | B(v) = B(V_{aut})\}$:

Proposition 9. *Suppose that the economy is in a degenerate steady state such that $v = \hat{v}$ and $k = \hat{k}$ every period. If $\hat{v} > V_{min}$, then $F(\underline{z}, \tilde{k}) \geq \beta\mathbb{E}[F(z, \tilde{k}) - F(z, 0)]$, where \tilde{k} is as defined in Proposition 8.*

The condition that $\hat{v} > V_{min}$ implies that the steady state is on the interior of the Pareto frontier. This is always the case with risk aversion, but we could not show it in general if utility is linear and the government is impatient. In short, the two propositions imply that $F(\underline{z}, \tilde{k}) \geq \beta\mathbb{E}[F(z, \tilde{k}) - F(z, 0)]$ is a sufficient condition for a degenerate steady state, and is a necessary condition for a degenerate steady state on the interior of the Pareto frontier.

The case of risk neutrality highlights the role that debt and consumption smoothing play in the long run dynamics of the benchmark economy. In particular, to the extent feasible, the optimal allocation with linear utility avoids movements in investment by keeping debt constant. This is feasible as long as shocks are not too dispersed. However, with dispersed shocks, the non-negativity constraint on consumption, coupled with impatience, generates long run dynamics reminiscent of the risk averse case. On the other hand, when $\beta(1+r) = 1$ assets are accumulated, so that consumption is eventually equalized across states in the steady state, and hence is positive in every state.

5.2 Balanced Budget

The central role that debt dynamics play in the benchmark model raises the question of how the economy behaves if the government runs a balanced budget. In particular, suppose that all transfers are financed through current taxes: $T(z^t) = \tau(z^t)\Pi(z^t) + g(z^t), \forall t, z^t$. Analyzing this case serves two purposes. It highlights the importance of debt in generating investment cycles in the benchmark economy. Secondly, budgetary restrictions have recently been considered by countries such as Brazil and Chile. The balanced budget analysis sheds light on the implications of such policies for the cyclical behavior of investment and consumption. In this subsection we briefly discuss the balanced budget case. For a more detailed analysis and formal proofs, see the working paper version of this paper (Aguiar et al., 2007).

Recall that in the benchmark model, the state variable is promised utility, which keeps track of debt commitments. If the government runs a balanced budget, debt is constant and there is no state variable linking periods. Optimal policies are therefore invariant to the realized history of shocks. Consequently, in the case with limited commitment, while investment may be distorted, there will be no cyclicity of investment. This is the case regardless of the relation between β and $(1+r)$.

Imposing a balanced budget has a second, more subtle implication. The absence of debt collapses the dynamic game between the government and foreign capitalists into an infinitely repeated stage game. The folk theorem implies that the full commitment allocation is attainable if the government is “patient enough,” where full commitment refers to the optimal allocation subject to budget balance. This allocation has first-best investment and constant consumption across all histories. Specifically, the folk theorem implies that there exists a $\beta^* \in (0, 1)$ such that for all $\beta \geq \beta^*$ the full commitment allocation is sustainable, and it is not sustainable for $\beta < \beta^*$.

Note that under a balanced budget, first best investment is sustainable either immediately or it is never sustainable. This contrasts with the benchmark case, in which the first best was approached in the limit if $\beta \geq 1/(1+r)$, after the government accumulated sufficient foreign assets. Moreover, depending on parameters, it may be the case that $\beta^* < 1/(1+r)$, implying that a balanced budget leads to higher capital and improved risk sharing than in the case with debt. In this case, access to sovereign debt markets generates cyclical and distorted investment and volatile consumption. If the government and the worker share the same discount factor, i.e. the government is benevolent, this volatile and distorted investment is associated with a welfare improvement, as the government optimally trades away future stability for higher consumption today. Placing a balanced budget constraint on a benevolent government can never be welfare improving.

However, if the government is impatient relative to domestic agents, it will not implement

the optimal allocation if it has access to debt. A balanced budget restriction may then improve domestic agents' welfare. In particular, if the domestic agents discount at the world interest rate, the allocation with first best investment and constant consumption is the first best solution to the domestic agents' unrestricted problem. That is, it is the allocation domestic agents would choose if they have full commitment and access to complete markets. This allocation can be achieved under limited commitment by imposing a balanced budget restriction on a government with a discount factor $\beta \geq \beta^*$.

6 Conclusion

The limited ability of the government to commit and the higher impatience of the government relative to the market are important features of developing markets. In this paper we showed that the combination of these features significantly alters the conclusions of the existing literature on sovereign debt and foreign direct investment. The long-run in this economy is characterized by distorted investment and investment cycles that prolong the effect of *i.i.d.* shocks. The analysis emphasized the distorting effect that sovereign debt has on investment. If a government is patient, debt is reduced (or assets accumulated) until the first best is sustainable. However, if the government is impatient relative to the world interest rate, debt and capital oscillates indefinitely, with low endowments associated with low investment. This debt overhang effect is derived endogenously and debt relief is shown not to alter the long-run behavior of investment.

The paper also highlighted the role of risk-aversion and access to debt markets in generating the results. If agents are risk neutral, there is no incentive to smooth consumption over time. In such an environment, the government to the extent feasible minimizes movements in debt in order to eliminate cyclical distortions to investment. Similarly, if the government runs balanced budgets and shocks are *i.i.d.*, capital is stable. This highlights the role debt, impatience, and limited commitment each play in amplifying investment cycles.

While it is clear that imposing a balanced budget stabilizes capital, it does not necessarily improve welfare. Indeed, when the government is benevolent, imposing an additional constraint on the government is never welfare improving, despite the increased income stability. The government borrows because domestic agents are impatient and are willing to trade more consumption today in exchange for more volatility in the future. However, if the government is impatient relative to the domestic agents as well as the world interest rate, the government will sub-optimally (from the domestic agents' perspective) trade away future stability for increased current consumption. Depending on the magnitude of this distortion, this may provide a rationale for imposing a balanced budget constraint.

A Appendix

A.1 Proofs of Lemma 1 and Proposition 1

Proof of Lemma 1:

Proof. As the objective functions are the same, we need to show that the constraint sets are equivalent. Suppose that $c(z^t)$ and $k(z^t)$ satisfy constraints (2) through (5). Taking expectations of both sides of equation (7) from the initial information set, we have (suppressing labor in the production function):

$$\mathbb{E} \left[F(z_t, k(z^{t-1})) - c(z^t) - (r + \delta)k(z^{t-1}) \right] = \mathbb{E} \left[(1 + r)b(z^{t-1}) - b(z^t) \right], \quad (20)$$

where we have used $\mathbb{E} [(1 - \tau(z^t))F_k(z_t, k(z^{t-1}))k(z^{t-1})] = \mathbb{E} [(r + \delta)k(z^{t-1})]$. We can solve this first order difference equation forward, applying the No Ponzi condition, to obtain:

$$\mathbb{E} \left[\sum_{t=0}^{\infty} \frac{1}{(1 + r)^t} \left(F(z_t, k(z^{t-1})) - c(z^t) - (r + \delta)k(z^{t-1}) \right) \right] \geq (1 + r)b(-1). \quad (21)$$

To go the other way, suppose that $c(z^t)$ and $k(z^t)$ satisfy (9). Starting from $b(-1)$, construct a sequence of $b(z^t)$ from the law of motion:

$$b(z^t) = \sum_{z_t \in Z} \pi(z_t) \left[c(z^t) - F(z_t, k(z^{t-1})) \right] + (r + \delta)k(z^{t-1}) + (1 + r)b(z^{t-1}). \quad (22)$$

Given $b(z^t)$, $c(z^t)$ and $k(z^t)$, the $\tau(z^t)$ solve equation (7) at each history. From (22), these taxes satisfy (5). The derivation of (7) also verifies that this choice is consistent with conditions (2) through (4). \square

Proof of Proposition 1:

Proof. The proof of (i) through (iii) follows straightforwardly from the solution to (8). Let λ be the multiplier on the single budget constraint. The first order conditions from the optimization are

$$\beta^t (1 + r)^t U'(c(z^t)) = \lambda, \quad (23)$$

$$\sum_{z \in Z} \pi(z) F_k(z, k(z^t)) = r + \delta. \quad (24)$$

To prove (iv) we use the budget constraint, which holds with equality as $\lambda > 0$. Let c_t be consumption at time t , which is independent of the history of shocks by (ii). The budget

constraint (9) implies that at any history z^t , we have

$$(1+r)b(z^{t-1}) = \sum_{s=t}^{\infty} \frac{\mathbb{E}[F(z_s, k^*)] - (r+\delta)k^*}{(1+r)^{s-t}} - \sum_{s=t}^{\infty} \frac{c_t}{(1+r)^{s-t}}. \quad (25)$$

Note that the expectation of $F(z_s, k^*)$ is independent of history given that capital is constant at k^* and z is *iid*. Therefore, debt does not depend on the particular path of shocks. \square

A.2 Proof of Proposition 2

Proof. The proposition holds by inverting the Pareto frontier. Specifically, the government's problem in recursive form with state variable b is

$$V(b) = \max_{c(z), b(z), k} \mathbb{E}[u(c(z)) + \beta V(b(z))] \quad (26)$$

subject to,

$$\begin{aligned} \mathbb{E}[c(z) + (1+r)b] &= \mathbb{E}[F(z, k) - (r+\delta)k + b(z)] \\ u(c(z)) + \beta V(b(z)) &\geq U(F(z, k)) + \beta V_{aut}, \quad \forall z \in Z \\ V(b(z)) &\geq V_{aut}, \quad \forall z \in Z. \end{aligned}$$

The first constraint is the expected budget constraint, derived from (7), where we have substituted in the first order condition for foreign direct investment. The second and third constraint ensure participation. Note that optimality ensures the budget constraint binds with equality. Therefore, $V(b)$ is strictly decreasing and has an inverse. By definition, $B(v)/(1+r)$ is this inverse, i.e. $V(B(v)/(1+r)) = v$. Therefore, an allocation that solves (26) must also solve (12). The converse is true as long as the promise keeping constraint (13) binds. When (13) does not bind, $B(v)$ is flat as we increase v (see Figure 1). The domain of v on which $B(v)$ is flat cannot be part of the Pareto frontier and $B(v)$ does not solve (26). This implies that if promise keeping does not bind at v , then there are no b such that $V(b) = v$. That is, the constraint set of (26) is empty and $b \notin \mathbf{b}$. \square

A.3 Proof of Lemma 2

Proved in the main text.

A.4 Proof of Propositions 3 – 5

This subsection characterizes the solution to the Bellman equation. The main technical challenge stems from the fact that the constraint set is not in general convex. However, assumption 7 is sufficient to ensure convexity of the constraint. Standard techniques can then be used to prove concavity and differentiability of the value function and the associated uniqueness and continuity of policy functions. With these results in hand, Propositions 3 – 5 follow immediately.

We first show that

Lemma 4. *Any optimal allocation has $k \leq k^*$ after any history*

Proof. The proof is direct. Suppose that after some history $k > k^*$, then a reduction in k while keeping the corresponding consumption allocation constant, satisfies the participation constraints. Note that such a decrease in k increases the foreign return in the current period. So it cannot be optimal to have $k > k^*$. \square

Let T denote the operator associated with the Bellman equation (12). We replace the capital stock as a choice variable with $h \equiv \mathbb{E}zf(k) - (r + \delta)k$. Notice there is a one-to-one monotonic mapping between h and $k \in [0, k^*]$, which is the relevant range given the previous lemma. Denote $K(h)$ to be the mapping from h to k . Let $H = [K^{-1}(0), K^{-1}(k^*)]$ be the appropriate domain for the choice variable h . Correspondingly, we replace the original choice set Ω with Ω' to account for the change in variable. The Bellman operator can then be expressed as:

$$TB^n(v) = \max_{(u(z), \omega(z), h) \in \Omega'} \sum_{z \in Z} \pi(z) \left[h + g(z) - c(u(z)) + \frac{1}{1+r} B^n(\omega(z)) \right] \quad (27)$$

subject to

$$\begin{aligned} v &\leq \sum_{z \in Z} \pi(z) [u(z) + \beta \omega(z)] \\ U(F(z, K(h))) + \beta V_{aut} &\leq u(z) + \beta \omega(z), \quad \forall z' \in Z. \end{aligned}$$

Note that the operator defined by (27) is a contraction. The value function is therefore the unique fixed point of this operator. Note as well that assumption 7 implies that $U(F(z, K(h)))$ is convex in h . Therefore, the constraint set is convex. Moreover, the objective function is concave if B^n is concave. The operator maps the space of bounded, continuous, concave functions into itself. The fact that T is a contraction implies that the fixed point is bounded, continuous, and concave.

To prove differentiability, we will appeal to the Benveniste and Scheinkman theorem (see [Stokey et al., 1989](#), Theorem 4.10). However, to do so, we must prove that optimal policies are interior. We do so in the following sequence of lemmas.

We begin by proving that capital is always greater than zero. We do so in two steps. We first show that the autarkic allocation is never optimal. Then, we show that this implies that zero capital is never optimal.

Lemma 5. *The autarkic allocation is never optimal.*

Proof. In an autarkic allocation $h = 0$ and $u(z) = U(F(z, 0))$ at all histories. From this allocation consider the following perturbation. Increase h by Δh and increase $u(z)$ by $\theta U'(F(z, 0))\Delta h$ at all histories, where

$$\theta \equiv \frac{U'(F(\underline{z}, 0))}{U'(F(\underline{z}, 0)) + \frac{\beta}{1-\beta}\mathbb{E}U'(F(z, 0))} < 1$$

Note that we have chosen θ so that participation is satisfied. To see this, note that participation can be written as

$$u(z) + \frac{\beta\mathbb{E}u(z)}{1-\beta} \geq U(F(z, 0)) + \beta V_{aut}.$$

The increase Δh raises the outside option by $U'(F(z, 0))F_k(z, 0)K'(0)\Delta h$. The inverse function theorem states that $K'(h) = (\mathbb{E}F_k(z, K(h)) - (r+\delta))^{-1}$. As $h \rightarrow 0$, $\mathbb{E}F_k(z, K(h))K'(h) \rightarrow 1$. Therefore, the outside option evaluated at $h = 0$ increases by $U'(F(z, 0))\Delta h$. The left hand side increases by $\theta [U'(F(z, 0)) + \beta\mathbb{E}U'(F(z, 0))]/(1-\beta) \Delta h$. By the definition of θ , this increase is greater than or equal to $U'(F(z, 0))\Delta h$, implying that participation is satisfied at the new allocation. The perturbation increases the objective function by

$$\Delta h \frac{1+r}{r} (1 - \mathbb{E}c'(U(F(z, 0)))\theta U'(F(z, 0))).$$

Note that $c'(U(F(z, 0)))U'(F(z, 0)) = 1$. The fact that $\theta < 1$ implies that this feasible perturbation raises the objective. Therefore, the original allocation is not optimal. \square

We now show that

Lemma 6. *In an optimal allocation, capital is always strictly positive: $h(v) > 0$ for all v , where $h(v)$ denotes the optimal choice of h given a promised utility v .*

Proof. Suppose, to generate a contradiction, that $h(v) = 0$ is optimal. Consider the following perturbation. Increase $h(v)$ by Δh . Let Z' denote the set of z for which participation holds

with equality. To satisfy participation for $z \in Z'$, increase $u(z)$ by $U'(F(z, 0))F_k(z, 0)K'(0)\Delta h = U'(F(z, 0))\Delta h$. Leave the allocations for $z \notin Z'$ unchanged. This will not violate participation for a small increase in h . The change in the objective function from this perturbation is

$$\Delta h \left(1 - \sum_{z \in Z'} \pi(z) c'(u(z)) U'(F(z, 0)) \right)$$

The binding participation constraint for $z \in Z'$, implies that $u(z) \leq U(F(z, 0))$, given that $\omega(z) \geq V_{aut}$. So that $c'(u(z))U'(F(z, 0)) \leq 1$ for $z \in Z'$. Note that the objective function strictly increases if there exists a $z \notin Z'$ or if for some $z \in Z'$, $u(z) < U(F(z, 0))$.

We consider two possibilities in turn: (i) Either $v > V_{aut}$, or $v = V_{aut}$ and the promise keeping constraint is slack; and (ii) $v = V_{aut}$ and the promise keeping constraint holds with equality:

- (i) Suppose the promise keeping constraint is slack or $v > V_{aut}$. Then $\mathbb{E}(u(z) + \beta\omega(z)) > V_{aut} = \mathbb{E}(U(F(z, 0)) + \beta V_{aut})$, where the last equality follows from the definition of V_{aut} . Therefore, there must be at least one state for which participation is slack at $h = 0$. That is, there exists a $z \notin Z'$, implying the change in the objective is strictly positive.
- (ii) Suppose that the promise keeping constraint holds with equality at $v = V_{aut}$. That is, $\mathbb{E}u(z) + \beta\omega(z) = V_{aut} = \mathbb{E}U(F(z, 0)) + \beta V_{aut}$. Together with the fact that $u(z) + \beta\omega(z) \geq U(F(z, 0)) + \beta V_{aut}$, it follows that the participation constraints bind for all $z \in Z$, that is $Z' = Z$. In this case, optimality of the original allocation requires that $u(z) = U(F(z, 0))$ for all z (or else the perturbation would be an improvement). However, this implies that $\omega(z) = V_{aut}$ for all z , given the binding participation constraints. Therefore the original allocation is the autarkic allocation, contradicting lemma 5.

This completes the proof. □

The fact that k is always strictly positive implies the following corollary:

Lemma 7. *Let V_{min} be the highest v such that $B(V_{min}) = B(V_{aut})$. Then $V_{min} \in (V_{aut}, V^*)$, where $V^* \equiv U(F(\bar{z}, k^*)) + \beta V_{aut} \leq V_{max}$. For all $v \geq V_{min}$ the promise keeping constraint holds with equality.*

Proof. V_{min} exists by continuity of the value function B . Given that $h(V_{aut}) > 0$, the promise keeping constraint at V_{aut} is slack. This follows from the participation constraints. Consider the relaxed problem at $v = V_{aut}$ where the promise keeping constraint is ignored, and let $u(z)$ and $\omega(z)$ be the associated solution. Given the slackness of the promise keeping

constraint, this problem will deliver the same allocation as the original problem. Define \hat{v} to be $\mathbb{E}(u(z) + \beta\omega(z))$ which is greater than V_{aut} given that the promise keeping constraint is slack. Hence $B(\hat{v}) = B(V_{aut})$. Moreover, for $v \geq \hat{v}$ promise keeping holds with equality. It follows then that $V_{min} \geq \hat{v} > V_{aut}$.

To show that $V_{min} < V^*$, suppose not. Note that at $v = V^*$, an optimal allocation delivers $k = k^*$, $u(z) = u(z')$ and $w(z) = w(z')$ with $u(z) + \beta w(z) = v \geq U(F(\bar{z}, k^*)) + \beta V_{aut}$, with all participation constraints slack. This follows from the fact that the allocation $k = k^*$ and $u(z) + \beta w(z) = V^*$, $\forall z \in Z$, with $u(z) = u(z')$, satisfies all constraints at $v = V^*$ by definition. This allocation is also the optimal allocation ignoring participation constraints, and therefore is optimal for the original problem. If it were the case that $B(V_{aut}) = B(V^*)$, the allocation delivered at V^* is thus optimal also for all $v < V^*$ (it satisfies promise keeping and achieves the optimal value for the objective). So for all $v < V^*$ the promise keeping constraint is slack. Moreover, participation is slack for at least one z by the fact that promised utility is equalized across states, while the outside option is strictly increasing in z . It follows then that optimality requires $u(z) = U_{min}$. To see that this must follow, suppose that $u(z) > U_{min}$ at some z , which implies that $u(z) > U_{min}$ for all z given the fact utility is equalized across states in the optimal allocation at V^* . As promise keeping and participation is slack for at least one z , it is feasible to lower $u(z)$ without lowering capital. As this increases the objective function, optimality requires $u(z) = U_{min}$ for all z . From the promise keeping constraint it follows then that

$$\begin{aligned} U_{min} + \beta V_{max} &\geq \mathbb{E}(u(z) + \beta\omega(z)) \geq V^* \\ \Rightarrow U_{min} &\geq V^* - \beta V_{max} \geq (1 - \beta)V^* \\ \Rightarrow U_{min} &\geq (1 - \beta)U(F(\bar{z}, k^*)) + \beta \mathbb{E}U(F(z, 0)) \\ \Rightarrow U_{min} &> \mathbb{E}U(F(z, 0)), \end{aligned}$$

a contradiction. □

Having shown that the optimal choice of capital is greater than zero, we now also show that the optimal choice of utility is always below $U_{max} > V_{max} - \beta V_{min}$.

Lemma 8. *For any $v \in [V_{aut}, V_{max}]$, $u(z) < U_{max}$*

Proof. Suppose not, then for some v_0 and z , $u(z) \geq U_{max}$. This implies that $u(z) + \beta V_{min} > U(F(\bar{z}, k^*)) + \beta V_{aut}$, by the definition of U_{max} . The participation constraint is slack at z . Given this, it follows that $u(z') \geq u(z)$, or else a reduction in $u(z)$ and an increase in $u(z')$ is feasible and would increase the objective function. Hence, all participation constraints are slack at v_0 . Strict concavity of the objective thus implies that $u(z) = u(z')$. Given

that continuation values are above V_{aut} , the delivered utility to the country is greater than $U_{max} + \beta V_{aut} > V_{max}$ where the last inequality follows by the definition of U_{max} . The promise keeping constraint is therefore slack at v_0 . Moreover, promise keeping is also slack for all $v \in [v_0, V_{max}]$, as the promised value can be increased without affecting the optimal allocation. In particular, promise keeping is slack at V_{max} . This however contradicts lemma 7. \square

The next lemma states that either $u(z) > U_{min}$ for all z , or $B'(v) = 0$:

Lemma 9. *Suppose at a promised utility v the optimal allocation calls for $u(z) = U_{min}$ for some $z \in Z$. Then $B'(v) = 0$.*

Proof. By definition, $c'(U_{min}) = 0$. That is, it is costless to increase $u(z)$ at the margin if $u(z) = U_{min}$. As $B(v)$ is non-increasing, the optimal response to a small increase in promised utility leaves the objective function unchanged. Therefore $B'^+(v) = 0$. This, plus the fact that $B(v)$ is non-increasing and concave, implies that $B'^-(v) = 0$ as well. \square

Corollary 2. *For $v > V_{min}$, the optimal allocation has $u(z) > U_{min}$ for all z .*

Proof. By definition of V_{min} and concavity, we have $B'^-(v) < 0$ for $v > V_{min}$. From the previous lemma, this implies that $u(z) > U_{min}$ for all z . \square

We are now ready to prove differentiability of the value function.

Lemma 10. *The value function $B(v)$ is differentiable at all v .*

Proof. For $v \in [V_{aut}, V_{min})$, $B(v)$ is constant and therefore the derivative is zero. If at $v = V_{min}$, there exists a $u(z) = U_{min}$, then the derivative is zero at V_{min} as well by lemma 9. Therefore, in the remainder of the proof we consider the case in which $u(z) > U_{min}$ for all z given $v = V_{min}$.

To prove differentiability for $v \in [V_{min}, V_{max}]$, we appeal to the Benveniste-Scheinkman Theorem (see [Stokey et al., 1989](#), Theorem 4.10). In order to do so, we construct a concave, differentiable function $W(v)$ defined on a neighborhood $N(v_0)$ of $v_0 \in [V_{min}, V_{max}]$, with $W(v_0) = B(v_0)$ and $W(v) \leq B(v)$ for $v \in N(v_0)$. Let $(u(z), \omega(z), k)$ denote the optimal allocation at v_0 . Define $\Delta v = v - v_0$. Define $\Delta k = \Delta v / \mathbb{E}F_k(z, k)$ and $\Delta u(z) = F_k(z, k)\Delta k = (F_k(z, k) / \mathbb{E}F_k(z, k)) \Delta v$. Note that Δk and $\Delta u(z)$ are linear functions of v , given v_0 . Define

$$W(v) = \mathbb{E} \left[F(z, k + \Delta k) - (r + \delta)(k + \Delta k) - c(u(z) + \Delta u(z)) + \frac{1}{1 + r} B(\omega(z)) \right].$$

Note that by definition, $\Delta u(z)$ is such that promise keeping is satisfied and participation holds at the adjusted capital. To see that $W(v) \leq B(v)$ for $v \in N(v_0)$, we show that the

allocations behind $W(v)$ are feasible or are sub-optimal. By lemma 6, $k > 0$, so small Δk does not violate the non-negativity constraint on capital. Moreover, if $k = k^*$, then by lemma 4 a small increase in k will never lead to an improvement. Similarly, $u(z)$ is always interior, so small $\Delta u(z)$ are always feasible. Specifically, the fact that $u(z) < U_{max}$ follows from 8. The fact that $u(z) > U_{min}$ for $v > V_{min}$ is the corollary of lemma 9, and we have already discussed the case of $u(z) = U_{min}$ for some z following $v = V_{min}$. Therefore $W(v) \leq B(v)$ for small Δv , and $W(v_0) = B(v_0)$. Moreover, $W(v)$ is differentiable and concave in $v \in N(v_0)$. Therefore, $B(v)$ is differentiable at v_0 . \square

We now show that eventually the continuation values lie above V_{min} , and once above V_{min} stay there forever.

Lemma 11. *For all $v \in [V_{aut}, V_{max}]$, there exists at least one $z \in Z$ such that $\omega(z) > V_{min}$, and for $v \in (V_{min}, V_{max}]$, $\omega(z) > V_{min}$ for all $z \in Z$.*

Proof. If $\omega(z) \leq V_{min}$, it is costless at the margin to increase $\omega(z)$. Optimality therefore requires that $u(z) = U_{min}$ whenever $\omega(z) \leq V_{min}$. Otherwise, we could strictly improve the allocation by reducing $u(z)$ and raising $\omega(z)$. Corollary 9 then implies that $\omega(z) > V_{min}$ for all $z \in Z$ when $v \in (V_{min}, V_{max}]$, which is the last statement of the lemma.

Now suppose that $v \in [V_{aut}, V_{min}]$. To see that there exists at least one $z \in Z$ such that $\omega(z) > V_{min}$, suppose on the contrary that $\omega(z) \leq V_{min}$ for all z . It is costless in this case to increase all $\omega(z)$ at the margin. Therefore, capital can be increased at zero additional cost. Optimality then requires that $k = k^*$. As $u(z) = U_{min}$ if $\omega(z) \leq V_{min}$, participation requires $U_{min} + \beta\omega(\bar{z}) \geq U(F(\bar{z}, k^*)) + \beta V_{aut}$. However, $V_{min} \geq U_{min} + \beta V_{min} \geq U_{min} + \beta\omega(\bar{z})$, where the second inequality follows from the premise that $\omega(\bar{z}) \leq V_{min}$. Taken together, we have $V_{min} \geq U(F(\bar{z}, k^*)) + \beta V_{aut} = V^*$, which contradicts lemma 7. \square

We now show that the value function is strictly concave for $v \geq V_{min}$.

Lemma 12. *The value function $B(v)$ is strictly concave for $v \in [V_{min}, V_{max}]$.*

Proof. Consider v_1 and v_2 , both in $[V_{min}, V_{max}]$ with $v_1 \neq v_2$. Recall that for any $v \in [V_{min}, V_{max}]$, the promise keeping constraint holds with equality. Let (u_i, h_i) denote the infinite sequence of utils and h corresponding to the optimal allocations given promised value v_i , $i = 1, 2$. Therefore,

$$B(v_i) = \sum_{t \geq 1, z^t} \pi(z^t) \frac{1}{(1+r)^{t-1}} \left[h_i(z^{t-1}) + g(z_t) - c(u_i(z^t)) \right],$$

for $i = 1, 2$. Let $u_\alpha = \alpha u_1 + (1 - \alpha)u_2$, $h_\alpha = \alpha h_1 + (1 - \alpha)h_2$, and $v_\alpha = \alpha v_1 + (1 - \alpha)v_2$ for $\alpha \in (0, 1)$. The fact that promise keeping holds with equality and $v_1 \neq v_2$ requires that

$u_1(z^t) \neq u_2(z^t)$ under some history z^t . Linearity of the objective function with respect to h and strict convexity with respect to $c(u)$ therefore implies that the allocation (u_α, h_α) yields a payout strictly greater than $\alpha B(v_1) + (1 - \alpha)B(v_2)$. Moreover, the convexity of the constraint set implies that (u_α, h_α) satisfies all constraints for a promised value v_α . Therefore, (u_α, h_α) is feasible from v_α , implying $B(v_\alpha) > \alpha B(v_1) + (1 - \alpha)B(v_2)$. \square

Corollary 3. *Let $g^x(v)$ denote the optimal policy for $x = u(z), \omega(z)$, and h , given promised value v . Then $g^x(v)$ is single valued and continuous for $v \in [V_{aut}, V_{max}]$.*

Proof. That policies are single valued follows directly from the fact that we are maximizing a strictly concave objective function subject to a convex constraint set. The Theorem of the Maximum states that policies are upper hemi-continuous. Single valuedness then implies continuity. Note that all $g^{\omega(z)} \in [V_{aut}, V_{min}]$ are in a sense equivalent promises, as the delivered utility will be V_{min} . The uniqueness result therefore uses the convention in this range that promised utility corresponds to delivered utility. \square

Lemma 13. *For any v , there exists non-negative multipliers $(\gamma, \lambda(z))$ such that in an optimal allocation $(u(z), \omega(z), h)$ we have that*

$$\begin{aligned} c'(u(z)) &= \gamma + \frac{\lambda(z)}{\pi(z)} \\ B'(\omega(z)) &= -\beta(1+r) \left(\gamma + \frac{\lambda(z)}{\pi(z)} \right) \\ \mathbb{E}F_k(z, K(h)) - (r + \delta) &= \sum_z \lambda(z) U'(F(z, K(h))) z f_k(K(h)) \end{aligned}$$

with complementary slackness conditions

$$\begin{aligned} \lambda(z) \left(U(F(z, K(h))) + \beta V_{aut} - u(z) - \beta \omega(z) \right) &= 0 \\ \gamma \left(v - \sum_z \pi(z) (u(z) + \beta \omega(z)) \right) &= 0. \end{aligned}$$

The envelope condition is

$$B'(v) = -\gamma$$

Proof. The Lagrangian is

$$\begin{aligned}\mathcal{L} = & - \sum_z \pi(z) \left[h + g(z) - c(u(z)) + \frac{1}{1+r} B(\omega(z)) \right] \\ & + \sum_z \lambda(z) \left(U(F(z, K(h))) + \beta V_{aut} - u(z) - \beta \omega(z) \right) \\ & + \gamma \left(v - \sum_z \pi(z) (u(z) + \beta \omega(z)) \right) \\ & + \sum_z \eta(z) (U_{min} - u(z)) + \sum_z \mu(z) (\omega(z) - V_{max}) + \kappa (h - h^*)\end{aligned}$$

The last three terms correspond to the constraints $u(z) \geq U_{min}$, $\omega(z) \leq V_{max}$ and $h \leq \mathbb{E}A(z)f(k^*) - (r + \delta)k^* \equiv h^*$, respectively. Recall that we have already proved that the optimal allocation requires $u(z) < U_{max}$, $\omega(z) > V_{aut}$ and $h > 0$, and so the corresponding constraints on $u(z)$, $\omega(z)$, and h can be omitted.

The envelope condition follows directly. The first order conditions are:

$$\begin{aligned}c'(u(z)) &= \gamma + \frac{\lambda(z) + \eta(z)}{\pi(z)} \\ B'(\omega(z)) &= -\beta(1+r) \left(\gamma + \frac{\lambda(z)}{\pi(z)} \right) + (1+r)\mu(z) \\ 1 &= \sum_z \lambda(z) U'(F(z, K(h))) F_k(z, K(h)) K'(h) + \kappa\end{aligned}$$

The last condition can be written as

$$1/K'(h) = \sum_z \lambda(z) U'(F(z, K(h))) F_k(z, K(h)) + \kappa/K'(h)$$

where $1/K'(h) = \mathbb{E}F_k(z, K(h)) - (r + \delta)$. If $h = h^*$, then $1/K'(h) = \kappa/K'(h) = 0$ and $\lambda(z) = 0$. If $h < h^*$, then $\kappa = 0$. Hence $\kappa/K'(h) = 0$ and the last condition becomes

$$\mathbb{E}F_k(z, K(h)) - (r + \delta) = \sum_z \lambda(z) U'(F(z, K(h))) F_k(z, K(h))$$

We now show that $\eta(z) = 0$ and $\mu(z) = 0$. If $u(z) > U_{min}$, then $\eta(z) = 0$ by complementary slackness. If $u(z) = U_{min}$ then the fact that $c'(U_{min}) = 0$ and the first order condition for flow utility imply that $\gamma = \lambda(z) = \eta(z) = 0$. Hence $\eta(z) = 0$.

To show that $\mu(z) = 0$, note that for $v = V_{max}$, we have that $\lambda(z) = 0$, $K(h) = k^*$, $u(z) = \bar{u}$ and $\omega(z) = \bar{\omega}$ for some \bar{u} and $\bar{\omega}$ such that $\bar{u} + \beta\bar{\omega} = V_{max}$ and $B'(V_{max}) = -c'(\bar{u})$, where this last step follows from the envelope and the first order condition for flow utility together with $\lambda(z) = 0$.

Now, to generate a contradiction, suppose that for some v we have $\mu(z) > 0$. The fact that $\mu(z) > 0$ implies that $\omega(z) = V_{max}$ and

$$B'(V_{max}) - \beta(1+r)B'(v) = (1+r) \left(\mu(z) - \beta \frac{\lambda(z)}{\pi(z)} \right).$$

where we have used the envelope condition. Given concavity and the fact that $\beta(1+r) \leq 1$, it follows that $B'(V_{max}) - \beta(1+r)B'(v) \leq 0$ and so $\beta \frac{\lambda(z)}{\pi(z)} \geq \mu(z)$. And so $\mu(z) > 0$ implies $\lambda(z) > 0$. Note as well that $-c'(\bar{u}) \equiv B'(V_{max}) = -\beta(1+r)c'(u(z)) + (1+r)\mu(z)$. Again $\beta(1+r) \leq 1$, implies that $c'(\bar{u}) < c'(u(z))$ and $u(z) > \bar{u}$.

From the binding participation constraint:

$$u(z) + \beta V_{max} = U(F(z, K(h))) + \beta V_{aut}$$

but $u(z) + \beta V_{max} > \bar{u} + \beta \bar{w} = V_{max} = U(F(\bar{z}, k^*)) + \beta V_{aut} > U(F(z, K(h))) + \beta V_{aut}$. A contradiction. So $\mu(z) = 0$. \square

The preceding lemmas complete the proof of Proposition 3. Specifically, part (i) is lemma 10; part (ii) is lemma 7; part (iii) is lemma 12; and part (iv) is lemma 13 substituting back in $k = K(h)$.

Proof of Proposition 4

Proof. Part (i) is lemma 3. Part (ii) is lemma 4. Part (iii) follows from the first order conditions and the envelope condition, as well as the strict convexity of $c(u)$. For part (iv), note that the first order conditions imply that if utility varies across states or if $B'(\omega(z))$ varies across states, then at least one $\lambda(z) > 0$, hence the first order condition for capital implies that capital is below the first best. Part (v) is lemma 11. \square

We will use the following lemma for Propositions 5 and 7:

Lemma 14. *Let $\lambda(z|v)$ denote the multiplier on the participation constraint in state z given a promised value v . Suppose the optimal allocation for promised values v_1 and v_2 is such that $h(v_1) > h(v_2)$, then there exists at least one $z \in Z$ at which $\lambda(z|v_1) < \lambda(z|v_2)$.*

Proof. From the first order condition we have

$$1 = \sum_z \lambda(z|v_i) U'(F(z, K(h_i))) F_k(z, K(h_i)) K(h_i),$$

for $i = 1, 2$. Differencing and re-arranging, we have

$$0 = \sum_z [\lambda(z|v_1) - \lambda(z|v_2)] U'(F(z, K(h_1))) F_k(z, K(h_1)) K(h_1) \quad (28)$$

$$+ \sum_z \lambda(z|v_2) [U'(F(z, K(h_1))) F_k(z, K(h_1)) K(h_1) - U'(F(z, K(h_2))) F_k(z, K(h_1)) K(h_2)].$$

Assumption 7 implies that $U'(F(z, K(h))) F_k(z, K(h)) K(h)$ is increasing in h . That is, $U'(F(z, K(h_1))) F_k(z, K(h_1)) K(h_1) > U'(F(z, K(h_2))) F_k(z, K(h_2)) K(h_2)$, for all z . Moreover, $\lambda(z|v_2) \geq 0$ for all z , and the fact that $h_2 < h_1 \leq h^*$ implies that at least one $\lambda(z|v_2) > 0$. Therefore, the second term in the above expression is strictly positive. Therefore, this plus the fact that $U'(F(z, K(h_1))) F_k(z, K(h_1)) K(h_1) > 0$ implies there exists at least one z such that $\lambda(z|v_1) < \lambda(z|v_2)$. \square

Proof of Proposition 5

Proof. For (i), let $v_1 > v_2$. To generate a contradiction, suppose $h(v_1) < h(v_2)$. From Lemma 14, there exists at least one $z \in Z$, call it z' , such that $\lambda(z'|v_1) > \lambda(z'|v_2)$. Concavity of $B(v)$ implies $\gamma(v_1) \geq \gamma(v_2)$. The first order conditions then require $\omega(z'|v_1) > \omega(z'|v_2)$ and $u(z'|v_1) > u(z'|v_2)$. This implies that the total utility delivered in z' (that is, $u(z') + \beta\omega(z')$) is greater following v_1 than v_2 . The premise of the contradiction is that $h(v_1) < h(v_2)$, which implies that the participation constraint is easier to satisfy. Therefore, the participation constraint in state z' following (v_1) must be slack. This implies that $0 = \lambda(z'|v_1) \leq \lambda(z'|v_2)$, which is a contradiction.

We now rule out $h(v_1) = h(v_2)$ if $h(v_2) < h^*$. To generate a contradiction, suppose that $h(v_1) = h(v_2) < h^*$. By strict concavity, $\gamma_1 > \gamma_2$. From (28), $h(v_1) = h(v_2)$ requires $\lambda(z|v_1) = \lambda(z|v_2)$, for all $z \in Z$. To see this, if $\lambda(z|v_1) > \lambda(z|v_2)$ at some z , then the fact that $\gamma_1 > \gamma_2$ and the first order conditions imply that delivered utility is higher in z following v_1 than v_2 . As the capital is the same under the premise, this implies that participation is slack in state z following v_1 , or $\lambda(z|v_1) = 0 \leq \lambda(z|v_2)$, a contradiction. Therefore, the only way (28) can hold is for $\lambda(z|v_1) = \lambda(z|v_2)$, for all $z \in Z$. Therefore, the fact that $\gamma_1 > \gamma_2$ and the first order conditions imply that $u(z)$ and $\omega(z)$ is strictly greater in all $z \in Z$ following v_1 than following v_2 . This implies that $\lambda(z|v_1) = 0, \forall z \in Z$, which implies $h(v_1) = h^*$, a contradiction.

For (ii), the fact that $-B'(g^{\omega(z)}(v)) = \beta(1+r)c'(g^{u(z)})$ and concavity ensures that $g^{\omega(z)}$ moves one-for-one with $g^{u(z)}$. We therefore focus on $g^{u(z)}$. The first order condition (16) and the envelope condition imply $c'(g^{u(z)}) = -B'(v) + \lambda(z)/\pi(z)$. Strict concavity of B implies that $-B'(v)$ is strictly increasing. If $\lambda(z) = 0$, this proves the claim as $\lambda(z)$ cannot fall below

zero. If $\lambda(z) > 0$, then the participation constraint binds and $k < k^*$. Binding participation and the fact that k is strictly increasing in v if $k < k^*$ implies that $g^{u(z)}$ is strictly increasing in v .

For (iii), we first show that promised continuation values are non-decreasing in z . The fact that $U(F(z, K(h)))$ is strictly increasing in z implies that either $u(z_1) + \beta\omega(z_1) > u(z_0) + \beta\omega(z_0)$ for $z_1 > z_0$, or that $\lambda(z_0) = 0$. If the former, the fact that $c'(u(z)) = -\beta(1+r)B'(\omega(z))$ and strict convexity of $c(u)$, imply that $u(z_1) + \beta\omega(z_1) > u(z_0) + \beta\omega(z_0)$ requires $\omega(z_1) \geq \omega(z_0)$. In the case that $\lambda(z_0) = 0$, we have that $\lambda(z_1) \geq \lambda(z_0)$ and (17) plus the concavity of B gives the result.

We now show that $g^{\omega(\bar{z})}(v) > g^{\omega(\underline{z})}(v)$ for $v < V^*$. Consider the set $Z' \subseteq Z$ such that $\lambda(z) > 0$ if $z \in Z'$. As $v < V^*$ there exists at least one z such that $\lambda(z) > 0$, and therefore Z' is not empty. As $U(F(z, K(h))) + \beta V_{aut}$ is strictly increasing in z , then so is $u(z) + \beta\omega(z)$ for $z \in Z'$. As $c'(u(z)) = -B'(\omega(z))$ and the strict concavity of $B(v)$ on the relevant domain, we have that $\omega(z)$ is strictly increasing in z for $z \in Z'$. Moreover, the first order condition for $\omega(z)$ implies that $\omega(z') > \omega(z)$ if $z' \in Z$ and $z \in Z - Z'$. This implies that the distribution of $\omega(z)$ is not a single point. The fact that $\omega(z)$ is non-decreasing in z over the entire set Z then implies the result. \square

A.5 Proof of Proposition 7

Proof. We focus on the invariant distribution of v . The invariant distribution of k follows immediately from the policy function $g^k(v)$. The policy function $g^{\omega(z')}(v)$ and the transition function for z induce a first-order Markov process for v . As $g^{\omega(z')}(v)$ is continuous, the transition function has the Feller property (see [Stokey et al., 1989](#), Exercise 8.10). Theorem 12.10 of [Stokey et al. \(1989\)](#) implies that there exists an invariant distribution.

To show that any invariant distribution is bounded above by V^* : As $\beta(1+r) < 1$ and the participation constraints are slack at V^* by definition, we have $g^{\omega(z')}(V^*) < V^*, \forall z' \in Z$. As $g^{\omega(z')}(v)$ is monotonic in v , then $g^{\omega(z')}(v) < V^*, \forall v \in [V_{aut}, V^*]$. This proves that the invariant distribution lies below V^* . As $g^k(v)$ is a function of v , the invariant distribution of v generates a corresponding distribution for k . As $g^k(v)$ is increasing in v for $v < V^*$ and $g^k(V^*) = k^*$, this implies that invariant distribution of k lies below k^* .

The fact that the invariant distribution is non-degenerate follows from the above fact that elements of the invariant distribution are less than V^* and Proposition 5.

To show that the invariant distribution is unique, we prove that there exists a “mixing point” \tilde{v} and an $N \geq 1$ such that there is strictly positive probability that $v \geq \tilde{v}$ and strictly positive probability $v \leq \tilde{v}$ after N periods starting from any point in $[V_{min}, V_{max}]$. The

result then follows from Theorem 12.12 of [Stokey et al. \(1989\)](#). Define \underline{v} to be the highest $v \in [V_{aut}, V_{max}]$ such that $g^{\omega(\underline{z})}(v) = v$. That is, \underline{v} is the highest v at which the policy function for $\omega(\underline{z})$ crosses the 45 degree line. Such a \underline{v} exists as the policy function is continuous and maps $[V_{aut}, V_{max}]$ into itself and from (ii) we know $\underline{v} < V^*$. Define \bar{v} to be the smallest v such that $g^{\omega(\bar{z})}(v) = v$, that is the smallest fixed point of the policy function for \bar{z} . We now show that $\bar{v} > \underline{v}$, that is, any fixed point of the policy function for \bar{z} is greater than the fixed point of the policy function for \underline{z} . Suppose not. From Proposition 5, we know that $\bar{v} \neq \underline{v}$. Therefore, the premise implies that $\bar{v} < \underline{v}$. The fact that $k(v)$ is strictly increasing in v for $v < V^*$ implies that $k(\underline{v}) > k(\bar{v})$. From Lemma (14), this implies that for at least one $z' \in Z$ we have $\lambda(z'|\bar{v}) > \lambda(z'|\underline{v})$. Now at the fixed points of the policy functions, the first order conditions and the envelope conditions imply

$$\begin{aligned} -B'(\underline{v}) &= \frac{\beta(1+r)}{1-\beta(1+r)} \frac{\lambda(\underline{z}|\underline{v})}{\pi(\underline{z})} \\ -B'(\bar{v}) &= \frac{\beta(1+r)}{1-\beta(1+r)} \frac{\lambda(\bar{z}|\bar{v})}{\pi(\bar{z})}. \end{aligned}$$

By concavity and $\underline{v} > \bar{v}$, this implies $\lambda(\underline{z}|\underline{v})/\pi(\underline{z}) \geq \lambda(\bar{z}|\bar{v})/\pi(\bar{z})$. The fact that $u(z)$ is increasing in z (Proposition 5) implies that $\lambda(z|v)/\pi(z)$ are increasing in z and therefore $\lambda(\underline{z}|\underline{v})/\pi(\underline{z}) \geq \lambda(\underline{z}|\bar{v})/\pi(\bar{v})$ and $\lambda(\bar{z}|\bar{v})/\pi(\bar{z}) \geq \lambda(\bar{z}|\underline{v})/\pi(\underline{v})$, for all $z \in Z$. This implies that $\lambda(\underline{z}|\underline{v}) \geq \lambda(\underline{z}|\bar{v})$, $\forall z \in Z$, which contradicts the existence of a z' such that $\lambda(z'|\bar{v}) < \lambda(z'|\underline{v})$. Therefore, $\bar{v} > \underline{v}$. Select \tilde{v} to be the midpoint of the interval $[\underline{v}, \bar{v}]$. Iterating on the highest shock policy function starting from any v , a long enough but finite sequence of high shocks will result in $v \geq \tilde{v}$. Similarly, using the lowest shock policy function and starting from any v , a finite sequence of low shocks will bring v below \tilde{v} . Therefore, \tilde{v} is a mixing point, and a unique ergodic distribution follows. \square

A.6 Section 5 Proofs

Proof of Proposition 8

Proof. The problem with linear utility can be written

$$B(v) = \max_{c(z), \omega(z), h} \mathbb{E} \left[h + g(z) - c(z) + \frac{1}{1+r} B(\omega(z)) \right]$$

subject to

$$\begin{aligned}
v &\leq \mathbb{E}[c(z) + \beta\omega(z)] \\
F(z, K(h)) + \beta V_{aut} &\leq c(z) + \beta\omega(z), \quad \forall z \in Z \\
0 &\leq c(z), \quad \forall z \in Z,
\end{aligned}$$

where as before we let $h = \mathbb{E}A(z)f(k) - (r + \delta)k$ replace $k \in [0, k^*]$ as the choice variable and $K(h) = k$. By definition, $F(z, K(h))$ is a convex function of h . Standard arguments imply that $B(v)$ is bounded, non-decreasing, and concave. Concavity implies that $B(v)$ is differentiable almost everywhere, but perhaps not differentiable at all points.

We proceed by considering the relaxed problem, in which we ignore the non-negativity constraint on consumption. Let $B_R(v)$ denote the corresponding value function. Note that the optimal h and $c(z)$ are interior. The fact that $h > 0$ follows from lemma 6, which did not require concavity of the utility function. The $c(z)$ are interior by definition of the relaxed problem. An argument along the lines of lemma 10 implies that $B_R(v)$ is everywhere differentiable. Similarly, h is increasing in v , and strictly increasing if $v < V^*$ (that is, if $K(h) < k^*$). This implies that $B_R(v)$ is strictly concave for $v < V^*$. In particular, $B'_R(v) = 0$ for $v < V_{min}$, $B'_R(v) = -1$, for $v > V^*$, and $B'_R(v)$ is strictly decreasing in v for $v \in (V_{min}, V^*)$.

We now state the first order conditions of the relaxed problem. Let γ_R denote the multiplier on the promise keeping constraint and $\lambda_R(z)$ the multiplier on participation. The first order conditions are

$$\begin{aligned}
1 - \gamma_R &= \frac{\lambda_R(z)}{\pi(z)} \\
B'_R(\omega(z)) &= -\beta(1+r) \left(\gamma_R + \frac{\lambda_R(z)}{\pi(z)} \right) \\
\mathbb{E} \left[\left(1 - \frac{\lambda_R(z)}{\pi(z)} \right) F_k \right] &= r + \delta
\end{aligned}$$

Substituting the first condition into the second yields, $B'_R(\omega(z)) = -\beta(1+r)$. Let \tilde{v} be such that $B'_R(\tilde{v}) = -\beta(1+r)$. The fact that $\beta(1+r) < 1$ implies that $\tilde{v} \in (V_{min}, V^*)$ and therefore \tilde{v} is unique given strict concavity on this domain. In the relaxed problem, the continuation utility is always \tilde{v} .

Now consider the case in which promised utility is \tilde{v} . In this case, the envelope condition implies $B'_R(\tilde{v}) = -\gamma_R = -\beta(1+r)$. Therefore, the first and final first order conditions imply $\mathbb{E}F_k = (r + \delta)/(\beta(1+r))$. That is, $k = \tilde{k}$. Note that $\lambda(z) > 0$ for all z , so participation and promise keeping together imply $\tilde{v} = \mathbb{E}F(z, \tilde{k}) + \beta V_{aut}$. Solving for consumption, we have

$$c(z) = F(z, \tilde{k}) + \beta V_{aut} - \beta \tilde{v} = F(z, \tilde{k}) - \beta \mathbb{E}F(z, \tilde{k}) + \beta(1 - \beta)V_{aut}.$$

Note that this allocation will be optimal for promised utility \tilde{v} in the original problem (including the non-negative consumption constraints) if $c(z) \geq 0$ for all $z \in Z$. Using the above expression for consumption, $c(z) \geq 0$ if $F(z, \tilde{k}) \geq \beta \mathbb{E}F(z, \tilde{k}) - \beta(1 - \beta)V_{aut}$, or $F(z, \tilde{k}) \geq \beta \mathbb{E} [F(z, \tilde{k}) - F(z, 0)]$, where we have used $V_{aut} = \mathbb{E}F(z, 0)/(1 - \beta)$. This holds for all z by the premise of the proposition. Therefore, $B(\tilde{v}) = B_R(\tilde{v})$ and if the economy reaches a promised utility \tilde{v} it will stay there forever. Moreover, $B(v) \leq B_R(v)$ for all v , given the fact that B_R refers to the value function for the relaxed problem. It follows that $B'^+(\tilde{v}) \leq B'_R(\tilde{v}) = -\beta(1+r)$ and $B'^-(\tilde{v}) \geq -\beta(1+r)$. The fact that $B_R(v)$ is strictly concave at \tilde{v} implies that $B(v)$ must also be strictly concave at \tilde{v} . Therefore, \tilde{v} is the unique v such that $B'^+(v) \leq -\beta(1+r) \leq B'^-(v)$.

Now consider the first order conditions for the original problem for arbitrary promised utility v , letting γ , $\lambda(z)$, and $\mu(z)$, be the multipliers associated with promise keeping, participation, and non-negativity, respectively. The first order conditions are:

$$\begin{aligned} 1 - \gamma &= \frac{\lambda(z) + \mu(z)}{\pi(z)} \\ B'^+(\omega(z)) &\leq -\beta(1+r) \left(\gamma + \frac{\lambda(z)}{\pi(z)} \right) \leq B'^-(\omega(z)) \\ \mathbb{E} \left[\left(1 - \frac{\lambda(z)}{\pi(z)} \right) F_k \right] &= r + \delta \end{aligned}$$

If $c(z) > 0$ at some z , then $\mu(z) = 0$. This implies that $\lambda(z)/\pi(z) = 1 - \gamma$, and substituting into the second equation we have $B'^+(\omega(z)) \leq -\beta(1+r) \leq B'^-(\omega(z))$. From the above, this implies that $\omega(z) = \tilde{v}$. As $c(z)$ cannot equal zero at all points in all histories given promise keeping, with probability one there will be a history such that $v = \tilde{v}$, and the promised utility will stay at \tilde{v} with associated capital \tilde{k} thereafter. \square

Proof of Corollary 1

Proof. It follows directly by substituting $F(z, k) = f(k) + g(z)$ into the condition of Proposition 8. \square

Proof of Lemma 3

Proof. From the proof of Proposition 8 we know that $c(z) = F(z, \tilde{k}) - \beta \mathbb{E}F(z, \tilde{k}) + \beta \mathbb{E}F(z, 0)$. Plugging back into the Bellman evaluated at the stationary value we have that

$$B(\tilde{v}) = \mathbb{E} \left[F(z, \tilde{k}) - (r + \delta)\tilde{k} - c(z) \right] + \frac{1}{1+r} B(\tilde{v})$$

Solving out for $B(\tilde{v})$ while using the consumption function above and that $\mathbb{E}F_k(z, \tilde{k}) = (r + \delta)/(\beta(1+r))$, the result follows. \square

Proof of Proposition 9

Proof. Recall that the first order conditions for the linear problem evaluated at the steady state are:

$$\begin{aligned} 1 - \gamma &= \frac{\lambda(z) + \mu(z)}{\pi(z)} \\ B'^+(\hat{v}) &\leq -\beta(1+r) \left(\gamma + \frac{\lambda(z)}{\pi(z)} \right) \leq B'^-(\hat{v}) \\ \mathbb{E} \left[\left(1 - \frac{\lambda(z)}{\pi(z)} \right) F_k \right] &= r + \delta, \end{aligned}$$

where we have replace $\omega(z)$ with \hat{v} , as implied by the fact that \hat{v} is a steady state. The fact that \hat{v} is a steady state implies that at least one $c(z) > 0$. That is, $\mu(z) = 0$ for some z . Therefore, the first two conditions can be combined to imply that $B'^+(\hat{v}) \leq -\beta(1+r)$.

We first show that $\hat{k} \geq \tilde{k}$, where \tilde{k} is defined by $\mathbb{E}F_k(z, \tilde{k}) = (r + \delta)/(\beta(1+r))$. That is, $\mathbb{E}F_k(z, \hat{k}) \leq (r + \delta)/(\beta(1+r))$. To see this, consider the optimal allocation for a small increase in v from \hat{v} . One feasible response is to increase capital by $\Delta k = \Delta v / \mathbb{E}F_k(z, \hat{k})$, where $\Delta v = v - \hat{v} > 0$. To satisfy participation, increase $c(z)$ by $F_k(z, \hat{k})\Delta k$. Taking expectations, this satisfies promise keeping as well. The net return on this perturbed allocation is $-(r + \delta)\Delta k = -(r + \delta)/\mathbb{E}F_k(z, \hat{k})\Delta v$. As this perturbation is feasible, optimality implies that $B'^+(\hat{v}) \geq -(r + \delta)/\mathbb{E}F_k(z, \hat{k})$. As $B'^+(\hat{v}) \leq -\beta(1+r)$, we have $\beta(1+r) \leq (r + \delta)/\mathbb{E}F_k(z, \hat{k})$, or $\mathbb{E}F_k(z, \hat{k}) \leq (r + \delta)/(\beta(1+r))$.

We next show that at \hat{v} , the participation constraints bind with equality for all z . To see this, suppose that $\lambda(z') = 0$ for some $z' \in Z$, and we will generate a contradiction. That is, suppose $c(z') + \beta\omega(z') = c(z') + \beta\hat{v} > F(z', \hat{k}) + \beta V_{aut}$. Consider a small deviation in promised utility $\Delta < 0$. For small enough Δ , the change in the value function will be $B'^-(\hat{v})\Delta$. Now consider the following alteration to the optimal allocation at \hat{v} . Reduce $\omega(z')$ by $\Delta/(\pi(z')\beta)$, and keep all other allocations (including capital) the same. For small enough Δ , this satisfies

participation (and does not violate the lower bound for $\omega(z)$ of V_{aut} as $\hat{v} > V_{min} > V_{aut}$). Moreover, it yields a change in promised value of Δ , satisfying promise keeping. Therefore, it is a feasible, but perhaps not optimal, allocation at $\tilde{v} + \Delta$. The change in utility for the foreigner from this perturbation is $\frac{B'^-(\omega(z'))\Delta}{\beta(1+r)} = \frac{B'^-(\hat{v})\Delta}{\beta(1+r)}$. Optimality implies that

$$B'^-(\hat{v})\Delta \geq \frac{B'^-(\hat{v})\Delta}{\beta(1+r)}.$$

As $\Delta < 0$, we have

$$-B'^-(\hat{v}) \geq \frac{-B'^-(\hat{v})}{\beta(1+r)}.$$

As $\beta(1+r) < 1$, and $B'^-(\hat{v}) \neq 0$ as $\hat{v} > V_{min}$, we have

$$-B'^-(\hat{v}) \geq \frac{-B'^-(\hat{v})}{\beta(1+r)} > -B'^-(\hat{v}),$$

a contradiction.

Given that participation binds at every z given promised utility \hat{v} , we have $c(z) + \beta\hat{v} = F(z, \hat{k}) + \beta V_{aut}$. Taking expectation and applying promise keeping, we have $\hat{v} = \mathbb{E}[F(z, \hat{k}) + \beta V_{aut}]$. Substituting in at $z = \underline{z}$, we obtain $c(\underline{z}) + \beta\mathbb{E}[F(z, \hat{k}) + \beta V_{aut}] = F(\underline{z}, \hat{k}) + \beta V_{aut}$. Using $V_{aut} = \mathbb{E}F(z, 0)/(1 - \beta)$, this can be written:

$$c(\underline{z}) + \beta\mathbb{E}[F(z, \hat{k}) - F(z, 0)] = F(\underline{z}, \hat{k}). \quad (29)$$

As $c(\underline{z}) \geq 0$, we have

$$\beta\mathbb{E}[F(z, \hat{k}) - F(z, 0)] \leq F(\underline{z}, \hat{k}). \quad (30)$$

Considering the optimal allocation at \hat{v} , we consider two cases in turn: (i) $c(\underline{z}) > 0$ and (ii) $c(\underline{z}) = 0$. In the case of (i), $c(z) > 0$ for all z and so $\mu(z) = 0 \forall z \in Z$. The first order conditions imply that $\hat{k} = \tilde{k}$ and $\hat{v} = \tilde{v}$. Then equation (30) implies that $\beta\mathbb{E}[F(z, \tilde{k}) - F(z, 0)] \leq F(\underline{z}, \tilde{k})$, and the proposition holds.

Suppose now that (ii) $c(\underline{z}) = 0$. In this case, (29) implies that $\beta\mathbb{E}[F(z, \hat{k}) - F(z, 0)] = F(\underline{z}, \hat{k})$. Define

$$H(k) \equiv \beta\mathbb{E}[F(z, k) - F(z, 0)] - F(\underline{z}, k).$$

The proposition is true if $H(\tilde{k}) \leq 0$. Given that $H(\hat{k}) = 0$ and $\hat{k} \geq \tilde{k}$, it is sufficient to prove that $H'(k) \geq 0$ for all k . Note, however, that since $\mathbb{E}F(z, k) = \mathbb{E}[A(z)f(k) + g(z)]$, we have $H'(k) \geq 0 \Leftrightarrow \beta\mathbb{E}A(z) \geq A(\underline{z})$, which does not depend on k . Moreover, the fact that $F(\underline{z}, 0) > 0$ implies that $H(0) < 0$. The fact that $H(0) < 0$ and $H(\hat{k}) = 0$ implies that

$H'(k) > 0$ for all k . Therefore, $H(\tilde{k}) < 0$ and the proposition is proved. □

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